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THE ROLE OF ALGEBRA IN TOPOLOGY†

BY SOLOMON LEFSCHETZ

1. *Introduction.* The assertion is often made of late that all mathematics is composed of algebra and topology. It is not so widely realized that the two subjects interpenetrate so that we have an algebraic topology as well as a topological algebra.

The increasing importance of algebra in topology, a domain whose roots lie in a very different soil, signifies that here also the age of consolidation and coordination is upon us. My present object is primarily to show that a reasonable blend of the algebraic and topological points of view is possible. For this purpose I shall formalize my earlier procedure of immersion in variable euclidean spaces by means of which I introduced dual cycles into topology. We shall see that around this mode of attack it is possible to group many of the recent very interesting results of combinatorial topology.

I. Algebra of Complexes

2. *Abstract Complexes.* Abstract complexes have been investigated by various authors, notably in recent years by J. W. Alexander, W. Mayer, and A. W. Tucker.‡ While I shall lean particularly on Tucker's work, my discussion bears largely on simplicial complexes and their duals, the basic types in topology.

According to Tucker, then, an *abstract* complex K is a set of elements E, its cells, partially ordered relative to a transitive geometric relation of incidence $\prec$ ("on the boundary of"), and with certain additional (algebraic) relations of incidence to be described presently. Each E has a dimension p which is a positive or negative integer and shall be frequently denoted by an index, as E_p. Moreover $E \prec E'$ implies $\dim E < \dim E'$. A *p-chain* of K is a linear form

$$C_p = x_i E_p^i, \tag{1}$$

† An address delivered at Duke University, December 30, 1936, as the retiring presidential address, before the American Mathematical Society.

‡ J. W. Alexander, Transactions of this Society, vol. 28 (1926), pp. 301–329; W. Mayer, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 1–42, 219–258; A. W. Tucker, Annals of Mathematics, vol. 34 (1933), pp. 191–243.

where x_i is a member of a given additive abelian group $\mathfrak{G}$. There is attached to K a linear chain-function F, called a *boundary operator* and determined by

$$(2) \qquad F(E_p^i) = [E_p^i : E_{p-1}^j] E_{p-1}^j, \qquad F(x_i E_p^i) = x_i F(E_p^i),$$

where the *incidence-number* $[\ ]$ is an integer, which is $\neq 0$ only when $E_{p-1}^j < E_p^i$. The operator F is subjected to the condition

$$(3) \qquad FF = 0.$$

A chain Γ_p such that $F(\Gamma_p) = 0$, is a *p-cycle*. Owing to (3), any $F(C_{p+1})$ is a Γ_p. The difference (factor) group $G_p = \{\Gamma_p\} - \{F(C_{p+1})\}$ is the pth *homology* group of K over $\mathfrak{G}$.

If we replace everywhere E_p^i by $-E_p^i$, the relations just written remain true provided that all the incidence-numbers $[\ ]$ involving E_p^i are changed in sign. We agree that under these conditions K has not been changed, its cell E_p^i having merely been *reoriented*.

Dualization. Consider a new aggregate $K^* = \{E_i^p\}$, where $E_i^p \longleftrightarrow E_p^i$ is a (1-1) correspondence. We assume: (a) dim $E_i^p = -\dim E_p^i = -p$; (b) $E_i^p < E_j^q$ when and only when $E_q^j < E_p^i$; (c) $[E_j^{p-1} : E_i^p] = (-1)^p [E_p^i : E_{p-1}^j]$. It is a simple matter to verify that the resulting operator F for K^* satisfies (3); hence K^* is also a complex, the *dual* of K.

The changes in sign indicated under (c), which are convenient for some purposes, may be eliminated by reorienting the cells E_p^i, p even. Therefore the relation between K and K^* is in fact symmetrical, and in particular $K^{**} = K$. This symmetry lies at the root of all the duality properties in topology.†

There subsist between the homology groups G_p of K and G^p (for the dimension $-p$) of K^* the duality relations of the Poincaré or Pontrjagin types according to the choices of coefficient-groups. Thus if the coefficients are: integers for K, real numbers mod 1 for K^*, the groups G_p and G^p are one another's character-groups.

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† Some recent authors (Alexander, Kolmogoroff, Whitney) bring in this dualism by referring the properties of K^* back to K, through the transformation $E_i^p \to E_p^i$. As a consequence the operator F of K^* gives rise to a dual boundary operator F^* for K, with dim $F^*(C) = \dim C + 1$. The choice of procedure is largely a matter of taste.

3. *Simplicial Complexes.* The elements of a simplicial complex are the simplexes. An (abstract) *p-simplex* σ_p is any finite collection of $p+1$ elements or *vertices* $a_0, \cdots, a_p$; its *faces* are the proper subcollections of the set. σ is oriented by assigning an order to its vertices modulo an even permutation. A closed (abstract) simplicial complex is a collection $K=\{\sigma\}$ such that $\sigma \in K$ implies that every face of $\sigma \in K$. The incidence-numbers are determined by $[a\sigma:\sigma]=1$, under which (3) is verified.

A very suggestive description of K and its dual may be given in terms of a certain algebra $\mathfrak{A}$. To define it we introduce new symbols $a_i^{-1,l}$, $a_i^{-1,r}$, $+1$, -1, 0 and shall assume that all the symbols a other than 0, ± 1 obey the following rules of combination:

$$aa' = -a'a, \qquad a^2 = 0, \qquad a^{-1,l}a = 1;$$
$$a^{-1,r} = -a^{-1,l} \qquad (\text{hence } aa^{-1,r} = 1);$$

if neither a product $a \cdots a'$ nor its inverse is made up of the vertices of a $\sigma \in K$, it is to be set equal to zero.

The symbols $a^{-1,l}$, $a^{-1,r}$ play the role of left and right inverses. Usually only the former shall be considered and I shall merely write a^{-1} for $a^{-1,l}$. The only monomials that may occur are then of one of the types, $\pm\sigma = \pm a \cdots b$, $\pm\sigma^{-1} = \pm b^{-1} \cdots a^{-1}$. The products of any two of the monomials are obtained in the customary formal way and always give rise to monomials of one of the two types or to ± 1, 0. The associative law holds between products of σ's or of σ^{-1}'s alone, but not between products involving both types. Thus $(a^{-1,l}a)a^{-1,r} = a^{-1,r} \neq a^{-1,l}(aa^{-1,r}) = a^{-1,l}$. It follows in particular from these rules of combination that $(c^{-1} \cdots b^{-1}a^{-1})(a\,b \cdots c) = 1$. For this reason we designate the first factor by $(a\,b \cdots c)^{-1}$. This is again a left-handed inverse, the exponent -1 taking here also the place of $-1, l$.

Each oriented σ of K determines a unique monomial product of positive powers which we shall likewise denote by σ. The p-chains over a group of coefficients $\mathfrak{G}$ are the forms of degree p with coefficients in $\mathfrak{G}$. They are members of a ring $\mathfrak{R}$ (non-commutative and non-associative) whose elements are the polynomials in the base elements of $\mathfrak{A}$, with coefficients in $\mathfrak{G}$.

In terms of the inverse symbols, we have

$$(4) \qquad F(\sigma_p) = \sum a_i^{-1} \cdot \sigma_p \quad \text{for} \quad p > 0; \quad F(a_i) = 0.$$

Hence

$$F(C_p) = \sum a_i^{-1} \cdot C_p \quad \text{for} \quad p > 0; \quad F(C_0) = 0. \tag{5}$$

The direct verification of (3) is immediate.

Notice that if L is a closed subcomplex of K, and we consider only cells of $K-L$, setting all others equal to zero, the conditions for a complex are still verified. $K-L$ is a so-called *open* complex.

It is quite natural at this juncture to introduce the product σ_p^{-1} as an abstract $(-p)$-cell, the inverse of σ_p, and adopt the following rules: (a) $\sigma_p^{-1} \prec \sigma_q^{-1}$ when and only when $\sigma_q \prec \sigma_p$; (b) the boundary operator is as before

$$F(\sigma_p^{-1}) = (\sum a_i^{-1})\sigma_p^{-1} \quad \text{for every } p. \tag{6}$$

The verification of (3) is again immediate; hence $K^{-1} = \{\sigma^{-1}\}$ is a complex. The incidence relations and numbers show that it is isomorphic with K^*. In the future we may therefore replace the dual elements by inverses and K^* by K^{-1}.†

Augmented Complex. In certain applications it may be more convenient to define dim σ as equal to the number of vertices of σ. This may be done formally by considering 1 as a vertex of every σ, that is, by considering σ as being in effect 1σ. The boundary relations are then chosen as

$$F(C) = \sum a_i^{-1} \cdot C; \qquad F(x) = 0, \qquad x \in \mathfrak{G} \tag{7}$$

for every C. This is tantamount to the old rule wherein it took two points to make a zero-cycle, one being insufficient, whereas under our initial scheme every $F(a) = 0$. Under these assumptions we shall call K an *augmented* complex (see Tucker, loc. cit., p. 11).

Remark. Let $\mathfrak{R}$ be a Hausdorff space and $(a_0, \cdots, a_r)$ a finite covering of $\mathfrak{R}$ by open sets. The intersection diagram of the sets in the sense of Alexandroff is a simplicial complex K with the a's as vertices, the rule being that $a_{i_0} \cdots a_{i_p} \in K$ whenever the sets indicated have an intersection $\neq 0$. The incidences in K correspond to the inclusions $\supset$. Let us now designate by a_i^{-1} the

† Quotients of the positive symbols have been utilized by M. H. A. Newman (Cambridge Philosophical Transactions, vol. 27 (1931), pp. 491–504), in the treatment of combinatorial intersections.

closed set $\mathfrak{R}-a_i$. The aggregate $\{a_i^{-1}\}$ is merely any aggregate of non-intersecting closed sets. To $\prod a_i \neq 0$ there corresponds now $\prod(\mathfrak{R}-a_i^{-1}) \neq 0$ or $\sum a_i^{-1} \neq \mathfrak{R}$. It follows that the elements of the dual K^{-1} represent the mutual inclusions of the sums with $\prec$ corresponding this time to $\subset$. This gives a noteworthy geometrical interpretation for the duals of simplicial complexes. It may be pursued still further, in the construction of the projection-spectra, and leads to a natural interpretation for the opposite types of homomorphisms (inverse and direct) associated with the ordinary and the dual cycles of $\mathfrak{R}$.

II. Spheres

4. *Duality.* The duality properties of complexes and sets (duality theorem of Alexander and its extensions) are intimately connected with the topological properties of the complementary regions of their topological images in euclidean spaces or spheres. We may therefore expect to find a good approach to these questions in a "combinatorial" immersion of a given abstract complex into abstract spheres. This requires that we begin with a consideration of the latter.

The simplest abstract n-sphere H_n is the sum of the proper faces of a $\sigma_{n+1}=a_0 \cdots a_{n+1}$. We shall designate its dual by H_n^{-1}.

The complex H_n is the most elementary combinatorial n-manifold M_n. Owing to its simplicity we may take full advantage of the known theory of manifolds,† which is moreover easily developed for this special case. As in the general theory, the first step is to take the *barycentric subdivision* H_n' of H_n, and to relate duality to H_n'. The complex H_n' will have a single new vertex associated with each $a_{i_0} \cdots a_{i_p}$, and denoted by $(i_0 \cdots i_p)$.

We verify that the simplexes $\sigma_{n-p}^0=\sigma_p^{-1}\sigma_{n+1}$ have the mutual incidences of the duals σ_p^*. It follows that the aggregate $\{\sigma_p^{-1}\sigma_{n+1}\}=H_n^0$ is a complex isomorphic with the dual of H_n. Notice that σ_{n-p}^0 is uniquely determined by the condition $\sigma_p\sigma_{n-p}^0=(-1)^{p+1}\sigma_{n+1}$.

On the other hand if we consider for $\sigma_p=a_{i_0} \cdots a_{i_p}$ the sum σ_{n-p}^1 of the $(n-p)$-simplexes of H_n' with vertices $(i_0 \cdots i_p\ j \cdots k)$, we find that it is isomorphic with a barycentric sub-

† See Chapter III of my Colloquium Lectures, *Topology*, New York, 1930 (hereafter referred to merely as *Topology*).

division of type $\sigma_n^0{}_{-p}$. Therefore $\sigma_n^1{}_{-p}$ is essentially an $(n-p)$-simplex.† We shall agree to orient its component $(n-p)$-simplexes (of H_n') concordantly with the subdivision of $\sigma_n^0{}_{-p}$, and thus oriented we denote it by $(\sigma_p)^*$ or σ_p^*. The aggregate $\{\sigma^*\}$ is another isomorph of the dual of H_n, and it is this new complex which shall be designated temporarily by H_n^*.

Geometric Realization. The combination H_n, H_n^* is very similar to the well known configuration of a pair of regular tetrahedra symmetrical with respect to their common centroid. A more convenient but closely related geometric configuration is obtained as follows. Choose a geometric simplex $\bar{\sigma}_{n+1}$ in a euclidean space S_{n+1}, call G its centroid, and $\bar{\sigma}, \cdots$ the images in $\bar{\sigma}_{n+1}$ of the abstract elements $\sigma, \cdots$. The linear spaces joining G with the faces of $\bar{\sigma}_{n+1}$ decompose the boundary $\bar{H}_n$ into the elements of $\bar{H}_n'$. Let P, P' be the intersections with $\bar{H}_n$ of any line through G. Through the radial transformation $\bar{p}$: $P \to P'$, the simplexes of $(\bar{\sigma}_p^{-1} \cdot \bar{\sigma}_{n+1})'$ go into those of $(\bar{\sigma}_p)^*$. The latter is therefore an $(n-p)$-cell transverse to $\bar{\sigma}_p$ and intersecting it in its centroid. These facts follow at once from the expressions of the simplexes of $\bar{\sigma}_p^{-1\prime}$ and $(\bar{\sigma}_p)^*$. We observe also that the spaces $\bar{\sigma}$, $\bar{\sigma}^*$ are actually orthogonal, so that here both duality and "algebraic" orthogonality correspond to geometric orthogonality.

The geometric transformation $\bar{p}$ determines a simplicial transformation ρ: $H_n' \to (H_n^{-1})'$, hence also $H_n^* \to H_n^{-1}$, characterized by permuting the vertices of H_n' associated with σ_p and $\sigma_p^{-1}\sigma_{n+1}$.

5. *Chain-Products.* Generally speaking we shall understand by chain-product a chain-function $C \odot D$ of two chains C, D with the following properties:

POSTULATE I. *$C \odot D$ is a bilinear function of C and D;*

POSTULATE II. $\dim (C \odot D) = \dim C + \dim D$;

POSTULATE III. $F(C \odot D) = F(C) \odot D + (-1)^{\dim C} C \odot F(D)$.

The following noteworthy instances justify the definition.‡

† I am indebted for this remark and also for suggesting the radial transformation $\bar{p}$ below to one of my students, Mr. Shaun Wylie.

‡ The three types (a), (b), (c) of *Topology* were unified by Tucker into a single w-multiplication, differing only from the above in that the power of (-1) was $w + \dim C$, with a different w in each case.

(a) *Intersections of chains.* Let C^* denote the result of replacing in C every σ by its dual σ^*. We have developed (loc. cit.) a complete combinatorial theory for the intersections of chains D and C^*, written $D \cdot C^*$. Then our postulates are fulfilled with $C^* \odot D = D \cdot C^*$.

(b) *Euclidean-products.* This operation on two complexes is the abstract equivalent of the topological or euclidean product of their geometric realizations and plays an essential role in problems of fixed points (*Topology*, Chapter VI).

(c) *Joins of two simplicial complexes.* Let $K = \{\sigma\}, K' = \{\sigma'\}$ be two augmented complexes. Their join KK' is the augmented complex $\{\sigma\sigma'\}$. If C, D are chains of K, K', CD is a definite chain of KK', which as a function of both satisfies our three postulates.

(d) *The products of Čech and Whitney.*† They are reducible to intersections combined with certain piece-wise simplicial transformations. We are back to H_n, and adopt a fixed order for its vertices. We agree also that in any expression $a_i a_j \cdots a_k$, σ, $(i, j, \cdots, k), \cdots$, the indices are always to be taken in ascending order.

Now there is a simplicial transformation τ: $H_n' \to H_n$ defined by $(i, j, \cdots, k) \to a_k$. When the vertices in σa are consecutive beginning with a_0 we readily verify

$$\tau(\sigma a)^* = aF((\sigma a)^{-1}\sigma_{n+1}). \tag{8}$$

Consider now a product $\sigma a \sigma'$, where the vertices are also consecutive beginning with a_0 and let $\odot$ denote the intersection operation of (a). We verify:

$$\tau((\sigma a)^* \odot \sigma a \sigma') = a\sigma', \tag{9}$$

$$\tau[(a\sigma')^* \odot \tau(\sigma a)^*] = \tau(\sigma a \sigma')^*. \tag{10}$$

† The earlier product of Alexander-Kolmogoroff must be mentioned here. See notably Alexander, Annals of Mathematics, (2), vol. 37 (1936), pp. 698–708; Čech, ibid., pp. 681–697. The connection of these recent products with my intersection theory has just been carried out by Wylie by a scheme analogous to the one utilized below. It explains why they are so numerous and how they are interrelated. The Alexander-Kolmogoroff ring based on these products is identical with the ring obtained by Gordon (Annals of Mathematics, (2), vol. 37 (1936), pp. 519–525) on the basis of our topological intersection theory as proved by Freudenthal in a forthcoming paper in the Annals of Mathematics. Since writing the above, I have found that the methods and results of Freudenthal and Wylie are substantially similar.

If the indices of $\sigma a\sigma'$ are not consecutive but merely in increasing order, we again obtain (9) and (10) provided that τ is replaced by the analogous operation obtained when the order of the indices is: first those of $\sigma a\sigma'$, then the rest, in increasing order in each case. Finally in all other cases but those considered the operations performed at the left in (9) and (10) give zero for result.

We introduce now a new operation $\otimes$ defined in the following cases

$$(\sigma a)^{-1} \otimes \sigma a\sigma' = a\sigma', \tag{11}$$

$$(\sigma a)^{-1} \otimes (a\sigma')^{-1} = (\sigma a\sigma')^{-1}, \tag{12}$$

whenever the factors in $\sigma a\sigma'$ are in increasing order, while otherwise

$$\sigma^{-1} \otimes \sigma' = \sigma^{-1} \otimes \sigma'^{-1} = 0. \tag{13}$$

The products $C' = C^{-1} \otimes D$, and $C''^{-1} = C^{-1} \otimes D^{-1}$ are defined for all C and D by imposing Postulate I. The other two are readily verified.

The coefficients of the chains which are multiplied must be such that the operations indicated have meaning: namely members of a unique ring, or two additive groups between which multiplication is defined, and so forth.

Kronecker-Index. The intersection $\sigma \cdot \sigma^*$ is a single vertex of H_n' taken ± 1 times if σ^* is the dual of σ, $=0$ otherwise. The exact coefficient, ± 1, 0, is called the Kronecker-index of σ and σ^*, and designated by $(\sigma \cdot \sigma^*)$. From there we derive $(C_p \cdot C_p^*)$ by the additive property. The actual intersection is a zero-chain, and the index is the sum of its coefficients.

If we designate by $(C\ C')$ the constant term in the formal (algebraic) product of C and C', then

$$(\sigma \cdot (\sigma')^*) = (\sigma\sigma'^{-1}), \tag{14}$$

$$(C_p \cdot C_p'^*) = (C_p C_p'^{-1}). \tag{15}$$

Therefore the non-vanishing of the formal product (15) is a sufficient condition that C, C' have common p-simplexes.

6. *Application to Simplicial Complexes.* We return to our simplicial complex K with its vertices $a_0, \cdots, a_r$. We increase them by $a_{r+1}, \cdots, a_{n+1}$, n arbitrary, and construct H_n as before. The

cells $\sigma^*, \sigma \in K$, constitute this time a sub-complex of H_n^*, and it is this complex which we denote by K^*. As for the abstract dual of K, we designate it by K^{-1}. We find by direct verification that if we preserve everywhere merely the cells of K, K^*, and apply our operations only to these, then all the results obtained so far continue to hold. Thus in (9) we must have $\sigma a \sigma' \in K$, and in (10), we write zero at the right whenever $\sigma a \sigma'$ does not belong to K. The different abstract products of §5 will be defined as before with F as the proper operator for K and K^*, and for all multiplications to which it has been applied. We deduce directly from §5, Postulate III, that all chain-products have the following common properties:

A. *The product of two cycles is a cycle;*

B. *If one of the factor cycles in a product of two cycles is a boundary-cycle so is the product.*

Property B implies also that the homology class of the product of two cycles is uniquely determined by those of the factors. This is the basic property in the proof of the topological invariance or of the invariance under subdivision of intersection-cycles. It may of course be utilized for the same purpose for the other multiplications.

Manifolds. Manifolds may be fitted readily into our scheme. Roughly speaking if we wish to characterize K as a p-manifold, we merely need to require that if $\sigma \in K$, and $St(\sigma)$ is its star, then $St(\sigma) \cdot \sigma^* = \zeta$ is a p-circuit mod $K - St(\sigma)$ (*Topology*, Chapter III). It will follow that the intersections $\zeta \cdot \sigma_q^*$ determine $(p-q)$-chains which are related to the σ_q's like the duals. On the other hand they may also serve to determine the Betti groups and the like, and there follow the usual duality theorems for manifolds (*Topology*, Chapter III).

Geometric Realization. The notations being as in §4, we find by means of the symbolic expressions of the cells, that the simplexes of $\overline{K}^*$ make up the $\overline{H}_n'$-neighborhood of $\overline{K}$. If Φ is the boundary of the neighborhood then the cycles of $\overline{K}^*$ are those of this neighborhood mod Φ. Their homology relations and the like are those of $\overline{K}$ mod $\overline{H}_n - \overline{K}$. This confirms our assertion that the dual complex is tied up with those immersion properties of $\overline{K}$ in any n-sphere which are independent of n. The topological invariants of $\overline{K}^*$ (Betti groups and their invariants) are thus found to be topological properties of the complementary

domains of $\overline{H}_n - \overline{K}$, depending solely upon the topology of $\overline{K}$, but not upon n. The simplest of these invariant properties is Alexander's duality theorem for the Betti numbers of $\overline{K}$ and of $\overline{H}_n - \overline{K}$. If we introduce orthogonal groups of coefficients we have the corresponding extensions for the Betti groups due to Pontrjagin.

III. Correspondences and Transformations

7. *Product Complexes.* Let A, B be two complexes and T, U two transformations $A \to B$. What elements if any are transformed alike by T and U? This is essentially the problem of the coincidences. We have given coincidence and fixed point formulas for geometric complexes and certain abstract spaces (*Topology* Chapters VI, VII), and Tucker (loc. cit.) has completely treated the abstract complex. We shall consider the simplicial case only, where, owing to the negative dimensions, the treatment may be made particularly simple.

Let then A, B be two finite closed simplicial complexes and let a_i, b_j be their vertices. We augment both A and B so that the dimension of any simplex is equal to the number of its vertices.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be the algebras of the type of §3 corresponding to A, B, and let α_i, β_j be the monomials of $\mathfrak{A}$, $\mathfrak{B}$ describing the simplexes of A, B. We shall denote by α^i the left-inverse of α_i and by β^j the right-inverse of β_j, so that

$$\alpha^i \alpha_i = \beta_j \beta^j = 1, \qquad (i \text{ not summed}). \tag{16}$$

The elements α^i, β^j are those of the duals A^{-1}, B^{-1}.

We now introduce a new algebra $\mathfrak{AB}$, composed of $\mathfrak{A}$, $\mathfrak{B}$, whose base terms are $\alpha_i \beta^j$, $\alpha^i \beta_j$, ± 1, 0, the monomials indicated being zero unless each factor $\neq 0$. We verify under the circumstances that the aggregates

$$AB^{-1} = \{\alpha_i \beta^j\}, \qquad A^{-1}B = \{\alpha^i \beta_j\} \tag{17}$$

treated as if all the factors were simplexes, are dual to one anotheranother with $\alpha_i \beta^j$, $\alpha^i \beta_j$ as corresponding dual elements.†

† Our procedure may be made more "geometric" by completing A, B to spheres H_n^a, H_n^b, with $\sigma_n{}^a{}_{+1}$, $\sigma_n{}^b{}_{+1}$ as their simplexes, then taking the simplexes $\{\alpha_i \beta_j{}^{-1} \sigma_n{}^b{}_{+1}\}$, $\{\sigma_n{}^a{}_{+1} \alpha_i{}^{-1} \beta_j\}$ of the join $H^a H^b$. It may be observed that the chief advantage of our treatment of these questions over Tucker's is that practically no computations are necessary to show that various cell-aggregates encountered are complexes.

As a rule when we have a combination such as $\alpha_i\beta^j$, it shall be understood that dim $\alpha_i = -$dim β^j, and dim α_i shall be denoted by p.

8. *Correspondences.* By a correspondence $\mathfrak{C}_a^b$ between A and B^{-1} over a coefficient-ring $\mathfrak{G}$, is meant an aggregate of triples

$$\mathfrak{C}_a^b: \qquad \{\alpha_i, \beta^j, x_j^i\}, \qquad x_j^i \in \mathfrak{G}.$$

The numbers x may be thought of as weights with which the couples are taken.

$\mathfrak{C}_a^b$ determines the chain of AB^{-1}:

$$C_a^b = x_j^i \alpha_i \beta^j, \tag{18}$$

and also a dual correspondence between A^{-1} and B:

$$\mathfrak{C}_b^a: \qquad \{\alpha^i, \beta_j, \bar{x}_i^j\}, \qquad \bar{x}_i^j = x_j^i,$$

whose representative chain in $A^{-1}B$:

$$C_b^a = x_i^j \alpha^i \beta_j, \tag{19}$$

is dual to (18) except for possible change in sign. Notice that if x_p, $\bar{x}_p$ denote the matrices of the coefficients x, $\bar{x}$ for dim $\alpha_i = p$, then $\bar{x}_p = x_p'$.

Let $\mathfrak{D}_a^b$ be a second correspondence with D, y in place of C, x. We inquire whether $\mathfrak{C}_a^b$ and $\mathfrak{D}_a^b$ possess coincident couples, that is, whether some $(\alpha_i, \beta^j) \in \mathfrak{C}_a^b$ and also $\in \mathfrak{D}_a^b$. We shall then say that there is a *coincidence* of $\mathfrak{C}_a^b$ and $\mathfrak{D}_a^b$. A necessary and sufficient condition for such an occurrence is that

$$(-1)^p x_j^i \bar{y}_i^j = x_j^i \alpha_i \beta^j \cdot \bar{y}_i^j \alpha^i \beta_j \neq 0, \qquad (i, j \text{ unsummed}). \tag{20}$$

A sufficient condition for the occurrence of *some* coincidence is, from (5), that

$$-\Theta = (C_a^b D_b^a) \neq 0, \tag{21}$$

where (CD) represents the numerical term in this product, or the Kronecker-index of the chains. Θ is the algebraic number of coincidences. By actual computation

$$-\Theta = \sum (-1)^p x_i^j \bar{y}_j^i = \sum (-1)^p \text{ trace } x_p y_p'. \tag{22}$$

We might obviously assume that the coefficients x, y belong respectively to groups $\mathfrak{G}$, $\mathfrak{G}'$ between which a distributive multiplication is defined (for example, $\mathfrak{G}$, $\mathfrak{G}'$ are character groups of one another).

It is clear that the present result could only be of use in topology whenever Θ is expressible in topologically invariant terms. Since the only known chains with topological character are cycles, we are led to the following very natural assumption.

ASSUMPTION I. *The representative chains C_a^b, D_b^a are both cycles* (*Topology*, Chapter VI).

It is also expedient to add another assumption.

ASSUMPTION II. *The ring $\mathfrak{G}$ is the field of all rational numbers.*

Now the bases α, β for the chains of A and B^{-1} have been chosen "orthogonal" throughout, in the sense that

$$\alpha^i\alpha_j = \beta_i\beta^j = \delta_i^j \qquad \text{(the Kronecker } \delta\text{)}.$$

If we impose contragredient transformations upon the bases our results will be left formally unchanged.

Under the circumstances, if we introduce suitable canonical bases with the non-bounding cycles duly isolated, the only terms left in the expressions of the chains C, D will be products of such cycles. Returning now to the non-augmented complexes, with dimensions counted in the usual way, if X_i^j, Y_j^i, denote the coefficients in the transformation of the independent cycles, and their matrices are denoted by X_p, Y_p, we shall have

$$\Theta = \sum (-1)^p \text{ trace } X_pY_p' . \tag{23}$$

9. *Transformations.* A transformation T_{ab}: $A \to B$ over $\mathfrak{G}$ is a collection of homomorphisms between chain-groups determined by relations

$$T_{ab}: \qquad \alpha_i \to x_i^j\beta_j; \quad x \in \mathfrak{G}; \quad \dim \alpha_i = \dim \beta_j.$$

It determines a dual transformation T^{ab} and two dual correspondences $\mathfrak{C}_a^b$, $\mathfrak{C}_b^a$:

$$T^{ab}: \qquad \alpha^i = \bar{x}_j^i\beta^j, \qquad \bar{x}_j^i = x_i^j;$$

$$\mathfrak{C}_a^b: \qquad (\alpha_i, \beta^j; \bar{x}_j^i), \qquad \mathfrak{C}_b^a: \qquad (\alpha^i, \beta_j; x_i^j).$$

Let U_{ab} be a second T with $\mathfrak{D}$, y in place of $\mathfrak{C}$, x. A coincidence of T_{ab} and U_{ab} is a couple (α_i, β_j) such that $\beta_j \epsilon T_{ab}\alpha_i$ and also $\epsilon U_{ab}\alpha_i$. They correspond to the coincidences of $\mathfrak{C}_a^b$ and $\mathfrak{D}_a^b$.

Hence $\Theta \neq 0$ is also a sufficient condition for the existence of some coincidence of T_{ab} and U_{ab}. The matrices x_p, y_p are now the transformation matrices for the p-cells, and X_p, Y_p, those for the p-cycles. Here also restrictions are necessary. Now it follows from Assumption I that if α, β are cycles, then $C_a{}^b\beta$ and $\alpha D_b{}^a$ are both cycles. In order that this hold, we find that the following assumption is a necessary and sufficient condition.

ASSUMPTION I'. *T_{ab} and U^{ab} are permutable with the boundary operator F.*

This is Tucker's equivalent of our earlier Assumption I.

Suppose now that T, U are transformations of A into itself. We introduce another copy B of A, replace T, U by transformations $A \to B$, and then apply our formulas. In particular, if $U = 1$, we have $y_p = 1$, $Y_p = 1$, and hence

$$\Theta = -\sum (-1)^p \text{ trace } x_p = \sum (-1)^p \text{ trace } X^p. \tag{24}$$

The equality between the sums in (22), (23) is a so-called generalized Euler-Poincaré relation in the sense of Hopf and Tucker. When $T = U = 1$ it reduces in fact to the ordinary Euler-Poincaré relation.

10. *Coincidences and Fixed Points of Transformations of Geometric Complexes.* In formulas (23) and (24) one may readily recognize the abstract equivalent of (23), (28), (29) of *Topology*, Chapter VI, our early coincidence and fixed point formulas for geometric manifolds and complexes. Let us show how the present algebraic procedure is related to our former more topological method.

Let A be a closed simplicial complex, and A_0 a topological image of A in a space S_n. If $\{\gamma_p{}^i\}$ is a base for its rational p-cycles and C_{n-p} a cycle of S_n mod $S_n - A_0$, the relations

$$(\gamma_p{}^i \cdot C_{n-p}) = x_i, \qquad (x_i \text{ rational}),$$

uniquely determine a homology class for C_{n-p}. The aggregate of all these classes for all the possible choices of A_0, S_n, is the rational $(-p)$-*dual* cycle γ^p determined by the numbers x_i. We write $\gamma^p \sim 0$ when every $x_i = 0$ and set $(\gamma_p{}^i \cdot \gamma^p) = x_i$. These dual cycles (called pseudocycles in our earlier work), form an additive abelian group. Dual cycles and groups are thus introduced

directly without passing through the medium of the chains, by means of their "orthogonality" to the absolute cycles.

Suppose now that we have two geometric complexes A, B undergoing two transformations T, U into one another. By a *coincidence* of the two is meant a couple of points $x \in A, y \in B$, such that $y \in Tx, y \in Ux$. Does there exist an invariant analogous to Θ such that when it is $\neq 0$ there is at least one coincidence?

Let us identify A, B or subdivisions of them with geometric complexes which are subcomplexes of two euclidean regions Ω_n^a, Ω_n^b subdivided into simplexes of diameter $<\epsilon$ assigned. Let N_a, N_b be the Ω^a, Ω^b neighborhoods of A, B. Let us designate by α_i, β_i and α^i, β^i the p-cycles and $(-p)$-dual cycles of A, B. We assume them represented by obvious simplicial cycles.

Let S_{2n} be the product of the spaces of A and B. The chains $\alpha^i \times \beta_j$ and $\alpha_i \times \beta^j$ characterize two dual types of cycles of $S_{2n} - A \times B$, which we shall call Γ, Δ. Let us suppose that T, U may be extended to transformations $\overline{T}$, $\overline{U}$: $N_a \to N_b$, having the following properties:

(a) the transformations $A \to B$ which they determine are T, U; (b) the coincidences of $\overline{T}$, $\overline{U}$ are the same as those of T, U; (c) the sets in S_{2n} which describe $\overline{T}$, $\overline{U}$ coincide with two singular relative cycles Γ, Δ, of S_{2n} mod $S_{2n} - A \times B$, respectively expressible (in the sense of homology) in terms of the cycles of types $\alpha_i \times \beta^j$ and $\alpha^i \times \beta_j$. Under the circumstances the Kronecker-index $(\Gamma \cdot \Delta)$ is a suitable invariant Θ. For $(\Gamma \cdot \Delta) \neq 0$ implies that Γ, Δ intersect, hence that $\overline{T}$, $\overline{U}$ have a coincidence and by (b) this must hold regarding T, U. In fact if we express Γ, Δ respectively in terms of the cycles, $\alpha^i \times \beta_j$ and $\alpha_i \times \beta^j$, and apply my intersection theory (*Topology*, Chapter IV), we obtain precisely (23).

The simplest case is where T is a continuous single-valued transformation (=c.s.v.t.). We then choose N_a, N_b *normal* (*Topology*, p. 91). That is to say, for example, through every point P of $N_a - A$ there passes a unique projecting segment with endpoints Q on A and R on the boundary of N_a. We then choose $\overline{T}$ such that $\overline{T} \cdot QR = T \cdot Q$. It turns out that $\overline{T}$ has for image in S_{2n} a Γ. We choose for U a transformation such that U^{-1} is of the type of a c.s.v.t. and extend it to a $\overline{U}$ in a similar manner with the roles of A, B interchanged. The image of $\overline{U}$ is then a Δ.

Special case. B is a copy of A, U is $A \to B$ corresponding to the

identical transformation $A \to A$. Then $\overline{U}$ may be chosen of type Δ, and the number $(\Gamma \cdot \Delta)$ obtained is precisely (24).

Let us recall in concluding that the same formulas hold for transformations of compact metric HLC spaces. They are spaces endowed with a strong type of local connectedness in the sense of homology, analogous to that possessed by the so-called absolute neighborhood retracts.†

PRINCETON UNIVERSITY

CIRCLES IN WHICH $|F(x)|$ HAS A SINGULARITY OR ASSUMES PREASSIGNED VALUES

BY J. W. CELL

Let k be a given positive integer and let a_0 and $a_k \neq 0$ be two given constants. Let $F_k(x)$ be any member whatever of the class C_k of functions which are regular in the neighborhood of the origin and which there have the expansion

$$F_k(x) = a_0 + a_k x^k + a_{k+1} x^{k+1} + \cdots,$$

where a_0 and a_k are the two given constants.

THEOREM 1. *Let* $\eta(a_0, a_1) = 0$ *if* $|a_0| = 1$. *In case* $|a_0| < 1$, *let* $\eta(a_0, a_1) = \{1 - |a_0|^2\}/|a_1|$, *and if* $|a_0| > 1$, *let* $\eta(a_0, a_1) = \{2|a_0| \log |a_0|\}/|a_1|$. *Then in or on the circle* $|x| = \eta(a_0, a_1)$, *either* $F_1(x)$ *has a singularity or* $|F_1(x)|$ *assumes the value one. Moreover, no smaller radius will do for the whole class of functions* C_1.

COROLLARY. $\eta(a_0, 1) = |a_1| \ \eta(a_0, a_1)$.

PROOF. If $|a_0| = 1$, the theorem is granted, so we shall henceforth suppose that this is not the case. If $a_0 = re^{i\alpha}$, $(r \geqq 0)$, we define $E(x) = e^{-i\alpha} F_1(x)$. Then $|E(x)| = |F_1(x)|$ and hence we may, with no loss of generality in the proof, suppose that a_0 is real and non-negative.

CASE 1. $0 \leqq a_0 < 1$. There exists a positive number η such that for $|x| \leqq \eta$, $F_1(x)$ is regular and $|F_1(x)| < 1$. Now form

$$G(x) = \frac{F_1(x) - a_0}{-a_0 F_1(x) + 1}. \tag{1}$$

† See Duke Mathematical Journal, vol. 2 (1936), pp. 435–442.

This transformation maps $|F_1(x)|<1$ upon $|G(x)|<1$ and $|F_1(x)|=1$ upon $|G(x)|=1$. Hence when $|x|\leqq\eta$, $G(x)$ is regular and $|G(x)|<1$. We expand $G(x)$ in a power series which converges for $|x|\leqq\eta$ and obtain

$$G(x)=\frac{a_1x}{1-a_0^2}+\cdots. \tag{2}$$

We apply Cauchy's inequality for the first derivative and obtain $\eta\leqq(1-a_0^2)/|a_1|$.

To show that $\eta(a_0, a_1)$ cannot be less than this amount and hence that the equality sign must persist if this is to be true for every member of the class C_1, subject to the condition $0\leqq a_0<1$, it is sufficient to exhibit a particular member of C_1 which is regular for $|x|<(1-a_0^2)/|a_1|$ and which in this circle is less than one in absolute value. To this end we define a particular function $G(x)$ by the first term of the series (2) and thence a particular function $F_1(x)$ by the inverse of (1). This resulting function $F_1(x)$ is easily shown to have the requisite properties.

Let S be the region consisting of the interior and boundary of the circle $|x|=\eta(a_0, a_1)$. To complete the proof of the theorem for this case we must show that if $F_1(x)$ is regular throughout S, then $|F_1(x)|$ must assume the value one at least once in S. To that end we suppose that $F_1(x)$ is regular throughout S and that $|F_1(x)|$ is not equal to one at any point of S. Then, since $|F_1(0)|<1$, we would have $|F_1(x)|<1$ at every point of S. It would follow that $|G(x)|<1$ at every point of S. But by Cauchy's inequality,

$$|G'(0)|\leqq\frac{\max|G(x)|}{\eta(a_0, a_1)},$$

or by using the series (2),

$$\frac{1}{\eta(a_0, a_1)}\leqq\frac{\max|G(x)|}{\eta(a_0, a_1)}.$$

Hence $\max|G(x)|\geqq1$, which is a contradiction. Thus either $F_1(x)$ is not regular at every point of S, or else $|F_1(x)|\geqq1$ at some point of S. Now there exist members of the class C_1 which are regular for all values of x in S and hence the contradiction shows, since $|F_1(0)|<1$, that $|F_1(x)|=1$ at some point of S.

CASE 2. $1<a_0$. If $1<a_0$, there exists a positive number η such that for $|x| \leqq \eta$, $F_1(x)$ is regular and $|F_1(x)|>1$. We write

$$K(x) = \frac{\log F_1(x) - 1}{\log F_1(x) + 1}. \tag{3}$$

Let the region in the F_1 plane consisting of the exterior and boundary of the unit circle with center at the origin be the region R. If we use the principal determination for $\log F_1(x)$, this transformation maps R on a circular arc triangle with vertices on the unit circle in the K plane and in such a way that the boundary of R (the circumference of the unit circle and the point at ∞) maps upon part of the circumference of the unit-circle in the K plane and the remainder of R upon the interior of this circle. The map for a general determination of the logarithm is a similar circular arc triangle with the same properties as to the image of the boundary of R and of its interior. Moreover, these circular arc triangles fill up, without overlapping, the interior of the unit circle in the K plane.

If we agree to use the principal determination for $\log F_1(x)$, then for $|x| \leqq \eta$, $K(x)$ is regular and $|K(x)|<1$. Then $K(x)$ has the power series expansion $K(x)=\alpha_0+\alpha_1 x+\cdots$, where

$$\alpha_0 = \frac{\log a_0 - 1}{\log a_0 + 1}, \qquad \alpha_1 = \frac{2a_1}{a_0(\log a_0 + 1)^2}.$$

We may apply the results of our former case to $K(x)$. We form $(1-\alpha_0^2)/|\alpha_1|$ and obtain $(2a_0 \log a_0)/|a_1|$, which is $\eta(a_0, a_1)$ for this second case.

We replace a_0 by $|a_0|$ in our results and thus obtain the theorem as stated.

THEOREM 2. *Let $\beta \neq 0$ be a positive real number and let $\phi(a_0, a_1, \beta) = \eta(a_0/\beta, a_1/\beta)$. Then in or on the circle $|x| = \phi(a_0, a_1, \beta)$ either $F_1(x)$ has a singularity or else $|F_1(x)|$ assumes the value β.*

We define $L(x)=F_1(x)/\beta$ and apply Theorem 1 to $L(x)$ to prove this theorem.

THEOREM 3. *Let $\eta(a_0, a_k, k) = \{\eta(a_0, a_k)\}^{1/k}$. Then in or on the circle $|x| = \eta(a_0, a_k, k)$, either $F_k(x)$ has a singularity or else $|F_k(x)|$ assumes the value one.*

The proof of this theorem follows the same method as the proof of Theorem 1 with the exception that we now use Cauchy's inequality for the kth derivative instead of for the first derivative.

THEOREM 4. *Let* $\theta(a_0, a_1) = 0$ *if* $a_0 = 0$ *or if* $|a_0| = 1$. *Otherwise let* $\theta(a_0, a_1) = |\{2a_0 \log |a_0|\}/a_1|$. *Then in or on the circle* $|x| = \theta(a_0, a_1)$, *either* $F_1(x)$ *has a singularity or else* $|F_1(x)|$ *assumes the value zero or the value one.*

We may, in the proof, suppose that a_0 is positive and that $a_0 \neq 1$. If $0 < a_0 < 1$, we write

$$G(x) = \frac{\log F_1(x) + 1}{\log F_1(x) - 1},$$

and if $1 < a_0$,

$$H(x) = \frac{\log F_1(x) - 1}{\log F_1(x) + 1}.$$

In both cases the resulting function, for x interior to or on the boundary of an appropriate circle, is less than one in absolute value. We may then apply the result of the first case of Theorem 1 to complete the proof.

It is clear that theorems similar to 2 and 3 may be established as generalizations of Theorem 4. We remark that the existence of the various radii in these theorems is implied by the Picard-Landau theorems.*

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* See E. Landau, *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie*, 1929.

GENERALIZED CONVEX FUNCTIONS

BY E. F. BECKENBACH

1. *Introduction.* We shall be concerned in this paper with real finite functions $f(x)$ defined in an interval $a<x<b$. A function $f(x)$ is said to be *convex** in (a, b) provided, for an arbitrary subinterval (x_1, x_2) interior to (a, b), the curve $y=f(x)$ lies nowhere above the line segment joining its end points; that is, provided, for arbitrary x_1, x_2, x, with $a<x_1<x<x_2<b$,

$$f(x) \leqq F_{12}(x),$$

where $F_{12}(x)$ is the function of the form

$$F_{12}(x) = \alpha x + \beta$$

satisfying

$$F_{12}(x_1) = f(x_1), \qquad F_{12}(x_2) = f(x_2).$$

Several generalizations of the notion of a convex function to other classes of functions of one variable have found their way into the literature.† In what follows we discuss the notions which seem to underlie these classes of functions.

Let $F(x; \alpha, \beta)$ be a (two-parameter) family of real finite functions defined for $a<x<b$ and satisfying the following conditions:

(1) each $F(x; \alpha, \beta)$ is a continuous function of x;

(2) there is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a<x_1<x_2<b$, takes on arbitrary values y_1, y_2.

For example, such a simple $F(x; \alpha, \beta)$ as $x^2+\alpha x+\beta$ is not in-

* J. L. W. V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Mathematica, vol. 30 (1906), pp. 175–193.

† See E. Phragmén and E. Lindelöf, *Sur une extension d'un principe classique de l'analyse*, Acta Mathematica, vol. 31 (1908), pp. 381–406; G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Mathematische Zeitschrift, vol. 29 (1929), pp. 549–640; B. Jessen, *Über die Verallgemeinerungen des arithmetischen Mittels*, Acta Szeged, vol. 5 (1931), pp. 108–116; G. Valiron, *Fonctions convexes et fonctions entières*, Bulletin de la Société Mathématique de France, vol. 60 (1932), pp. 278–287.

cluded in any of the above mentioned generalizations. Other examples are $\alpha x+\beta$; $Ax^3+Bx^2+\alpha x+\beta$, where A and B are fixed constants; $\alpha e^{\rho x}+\beta e^{-\rho x}$, where ρ is a fixed constant; and, with $b-a \leqq \pi/\rho$, $\alpha \sin \rho x+\beta \cos \rho x$. Still another example of the family $F(x;\alpha,\beta)$ is the set of images of all non-vertical straight lines, under a one-to-one continuous transformation of the domain $a<x<b$ of the plane into itself in such a way that every vertical line is transformed into itself.

Members of the above family shall be denoted simply by $F(x)$, not $F(x;\alpha,\beta)$, individual members being distinguished by subscripts. In particular, $F_{ij}(x)$ shall denote the member satisfying

$$F_{ij}(x_i) = f(x_i), \qquad F_{ij}(x_j) = f(x_j), \qquad (a < x_i < x_j < b).$$

DEFINITION 1. The function $f(x)$ shall be called a *sub-$F(x;\alpha,\beta)$ function*, or simply a *sub-$F(x)$ function*, provided

$$f(x) \leqq F_{12}(x) \tag{3}$$

for all x_1, x_2, x, with $a<x_1<x<x_2<b$.

DEFINITION 2. *Super-$F(x)$ functions* are defined exactly as are sub-$F(x)$ functions, excepting that the sign of inequality is reversed; the analysis is the same, *mutatis mutandis*.

2. *The Family $F(x;\alpha,\beta)$.* We shall prove the following theorem.

THEOREM 1. *Let $a<x_0<b$, and let $F_r(x)$, $F_s(x)$ be two members of the family satisfying* (1) *and* (2) *such that*

$$F_r(x_0) = F_s(x_0), \tag{4}$$

$$F_r(x) \not\equiv F_s(x), \qquad (a < x < b); \tag{5}$$

then $F_r(x)>F_s(x)$ for all x in (a, b) on one side of x_0, while $F_r(x)<F_s(x)$ for all x in (a, b) on the other side of x_0.

PROOF. By (2), (4), and (5), $F_r(x)\neq F_s(x)$, $(a<x<b)$, except at x_0; consequently, by (1), on either side of x_0, one of $F_r(x)$, $F_s(x)$ is greater than the other. Suppose it could be the same one, say $F_s(x)$, which is greater on each side; we shall show this is impossible.

Let $a<x_1<x_0<x_2<b$, and consider $F_t(x)$, determined by

$$F_l(x_1) = F_s(x_1), \qquad F_l(x_2) = F_r(x_2).$$

Then $F_l(x_2) < F_s(x_2)$, so that $F_l(x) < F_s(x)$, $(x_1 < x < b)$; in particular,

$$F_l(x_0) < F_s(x_0). \tag{6}$$

Similarly,

$$F_l(x_0) > F_r(x_0). \tag{7}$$

Now (6) and (7) contradict (4), establishing the theorem.

COROLLARY 1. *If $a < x_1 < x_2 < b$, and if $F_r(x_1) < F_s(x_1)$, $F_r(x_2) < F_s(x_2)$, then $F_r(x) < F_s(x)$ for $x_1 \leqq x \leqq x_2$; but if $F_r(x_1) < F_s(x_1)$, $F_r(x_2) > F_s(x_2)$, then $F_r(x) < F_s(x)$ for $a < x \leqq x_1$, and $F_r(x) > F_s(x)$ for $x_2 \leqq x < b$.*

THEOREM 2. *Let the points (x_n, y_n), (x_n', y_n'), $(n = 0, 1, 2, \cdots)$, satisfy $a < x_n < x_n' < b$, and*

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0), \qquad \lim_{n\to\infty} (x_n', y_n') = (x_0', y_0'),$$

and let $F_n(x)$ be determined by

$$F_n(x_n) = y_n, \qquad F_n(x_n') = y_n'; \tag{8}$$

then

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad (a < x < b), \tag{9}$$

uniformly in any closed subinterval of $a < x < b$.

PROOF. For a given $\epsilon > 0$, and for an arbitrary $\bar{x}$ in (a, b), with $\bar{x} \neq x_0$, $\bar{x} \neq x_0'$, let the functions $F_p(x), \cdots, F_s(x)$ be determined by

$$F_p(x_0) = F_0(x_0), \qquad F_p(\bar{x}) = F_0(\bar{x}) + \epsilon, \tag{10, 11}$$

$$F_q(x_0) = F_0(x_0), \qquad F_q(\bar{x}) = F_0(\bar{x}) - \epsilon, \tag{12, 13}$$

$$F_r(x_0') = F_p(x_0'), \qquad F_r(\bar{x}) = F_p(\bar{x}) + \epsilon = F_0(\bar{x}) + 2\epsilon, \tag{14, 15}$$

$$F_s(x_0') = F_q(x_0'), \qquad F_s(\bar{x}) = F_q(\bar{x}) - \epsilon = F_0(\bar{x}) - 2\epsilon. \tag{16, 17}$$

Let δ denote the least of the four (minimum) distances from the curves $y = F_r(x)$ and $y = F_s(x)$ to the points (x_0, y_0) and (x_0', y_0'). It is easily verified, by Theorem 1 and (10)-(17), that $\delta > 0$. Let

$$\delta^* = \min\,(\delta,\ |\ \bar{x} - \bar{x}_0\ |,\ |\ \bar{x} - x_0'\ |); \tag{18}$$

then $\delta^* > 0$.

Now let $n_0 = n_0(\bar{x})$ be so large that

$$\begin{aligned} \{(x_n - x_0)^2 + (y_n - y_0)^2\}^{1/2} &< \delta^*, \\ \{(x_n' - x_0')^2 + (y_n' - y_0')^2\}^{1/2} &< \delta^*, \qquad (n \geqq n_0). \end{aligned} \tag{19}$$

We shall prove that

$$|\ F_n(\bar{x}) - F_0(\bar{x})\ | < 2\epsilon, \qquad (n \geqq n_0). \tag{20}$$

Three cases arise, as follows.

CASE 1. $x_0 < \bar{x} < x_0'$. In this case, by (10), (11), and Theorem 1, $F_p(x_0') > F_0(x_0')$; consequently, by (10), (14), (15), and Theorem 1, $F_r(x_0) > F_0(x_0)$, $F_r(x_0') > F_0(x_0')$. Similarly, $F_s(x_0) < F_0(x_0)$, $F_s(x_0') < F_0(x_0')$. By (1) and (19), then, for $n \geqq n_0$,

$$\begin{aligned} F_s(x_n) < y_n = F_n(x_n) < F_r(x_n), \\ F_s(x_n') < y_n' = F_n(x_n') < F_r(x_n'), \end{aligned}$$

whence it follows from Corollary 1 that

$$F_s(x) < F_n(x) < F_r(x), \qquad (x_n < x < x_n',\ n \geqq n_0). \tag{21, A}$$

In particular,

$$F_s(\bar{x}) < F_n(\bar{x}) < F_r(\bar{x}), \qquad (n \geqq n_0). \tag{22}$$

Now (20) follows from (15), (17) and (22).

CASE 2. $x_0' < \bar{x} < b$. The above analysis gives this time

$$F_r(x_0) < F_0(x_0) < F_s(x_0), \qquad F_r(x_0') > F_0(x_0') > F_s(x_0'),$$

and therefore, for $n \geqq n_0$,

$$\begin{aligned} F_s(x_n) > y_n = F_n(x_n) > F_r(x_n), \\ F_s(x_n') < y_n' = F_n(x_n') < F_r(x_n'), \end{aligned}$$

whence it follows from Corollary 1 that

$$F_s(x) < F_n(x) < F_r(x), \qquad x_n' < x < b, \qquad (n \geqq n_0). \tag{21, B}$$

In particular, (22), and therefore (20), hold in this case.

CASE 3. $a < \bar{x} < x_0$. The same analysis gives now

$$F_r(x_0) > F_0(x_0) > F_s(x_0), \qquad F_r(x_0') < F_0(x_0') < F_s(x_0'),$$

and therefore, for $n \geqq n_0$,

$$F_s(x_n) < y_n = F_n(x_n) < F_r(x_n),$$
$$F_s(x_n') > y_n' = F_n(x_n') > F_r(x_n'),$$

whence

$$(21, C) \qquad F_s(x) < F_n(x) < F_r(x), \qquad (a < x < x_n; n \geqq n_0),$$

so that again (22) and (20) hold.

For each x in (a, b), except for x_0, x_0', (9) follows from (20). Let x_t, x_t', with $x_t < x_t'$, be any other two points in (a, b). Then the sequences

$$(23) \qquad (x_t, F_n(x_t)), \quad (x_t', F_n(x_t')), \qquad (n = 1, 2, 3, \cdots),$$

satisfy, by the above analysis,

$$(24) \qquad \lim_{n\to\infty} (x_t, F_n(x_t)) = (x_t, F_0(x_t)),$$
$$\lim_{n\to\infty} (x_t', F_n(x_t')) = (x_t', F_0(x_t')).$$

Now since (24) holds, the above analysis can be applied to the sequences (23), giving (20) this time for $\bar{x} \neq x_t$, $\bar{x} = x_t'$, in particular for $\bar{x} = x_0$ and for $\bar{x} = x_0'$. Therefore (20) and (9) hold throughout (a, b).

In each of the above three cases, by (1), there is an interval $\bar{x} - \eta < x < \bar{x} + \eta$, where $\eta = \eta(\bar{x}) > 0$, in which

$$F_s(x) > F_s(\bar{x}) - \epsilon, \quad F_r(x) < F_r(\bar{x}) + \epsilon, \quad |F_0(x) - F_0(\bar{x})| < \epsilon,$$

and therefore, by (15), (17), and (21, A, B, C), in which

$$(25) \qquad |F_n(x) - F_0(x)| < 4\epsilon,$$
$$(\bar{x} - \eta(\bar{x}) < x < \bar{x} + \eta(\bar{x}); n \geqq n_0(\bar{x})).$$

The same discussion holds, according to the preceding paragraph, for intervals about x_0, x_0'. Then (25) holds for arbitrary $\bar{x}$, with $a < \bar{x} < b$, whence the uniformity of (9) for any closed subinterval of $a < x < b$ follows from a simple application of the familiar Heine-Borel Theorem.

COROLLARY 2. *Any subset of the family $F(x; \alpha, \beta)$ is compact, provided the ordinates are bounded for two distinct abscissas.*

PROOF. One readily sets up sequences satisfying the conditions of Theorem 2.

3. *Some Properties of Sub-$F(x;\alpha,\beta)$ Functions.*

THEOREM 3. *If $f(x)$ is a sub-$F(x)$ function, then for any x_1, x_2, with $a<x_1<x_2<b$, the inequality $f(x)\geqq F_{12}(x)$ holds for $a<x<x_1$, and for $x_2<x<b$.*

PROOF. Fix x_3, with $x_2<x_3<b$. Since

$$F_{13}(x_1)=F_{12}(x_1)=f(x_1),\qquad F_{13}(x_2)\geqq F_{12}(x_2)=f(x_2),$$

it follows from Theorem 1 that

$$f(x_3)=F_{13}(x_3)\geqq F_{12}(x_3).$$

A similar proof holds for $a<x_2<x_1$.

THEOREM 4. *If $f(x)$ is a sub-$F(x)$ function, and if, for some x_1, x_2, x_3, with $a<x_1<x_3<x_2<b$,*

$$f(x_3)=F_{12}(x_3), \tag{26}$$

then $f(x)\equiv F_{12}(x)$, $(x_1\leqq x\leqq x_2)$.

PROOF. By (2) and (26),

$$F_{12}(x)\equiv F_{13}(x)\equiv F_{32}(x); \tag{27}$$

consequently, by Theorem 3 and (27),

$$F_{12}(x)\equiv F_{13}(x)\leqq f(x),\qquad (x_3\leqq x<b), \tag{28}$$

$$F_{12}(x)\equiv F_{32}(x)\leqq f(x),\qquad (a<x\leqq x_3). \tag{29}$$

But according to (3),

$$F_{12}(x)\geqq f(x),\qquad (x_1\leqq x\leqq x_2). \tag{30}$$

The theorem follows from inequalities (28), (29), and (30).

THEOREM 5. *If $f(x)$ is a sub-$F(x)$ function, and if $a<x_3\leqq x_1<x_2\leqq x_4<b$, then*

$$F_{34}(x)\geqq F_{12}(x),\qquad (x_3\leqq x\leqq x_4). \tag{31}$$

PROOF. By Theorem 3,

$$F_{34}(x_3)=f(x_3)\geqq F_{12}(x_3),\qquad F_{34}(x_4)=f(x_4)\geqq F_{12}(x_4),$$

whence, by Theorem 1 and its corollary, (31) follows. Incidentally, by (2) and Theorem 1, the sign of equality in (31) holds nowhere in $x_3 \leqq x \leqq x_4$, except at most at one of the end points, unless it holds identically.

THEOREM 6. *If $f(x)$ is a sub-$F(x)$ function, then $f(x)$ is continuous.*

PROOF. We shall show that $f(x)$ is continuous at an arbitrary x_0, with $a<x_0<b$. Let $a<x_1<x_0<x_2<b$. Then, by Definition 1 and Theorem 3, for $x_1<x_0-h<x_0+h<x_2$,

$$F_{10}(x_0 - h) \geqq f(x_0 - h) \geqq F_{02}(x_0 - h),$$

$$F_{10}(x_0 + h) \leqq f(x_0 + h) \leqq F_{02}(x_0 + h).$$

Let $h \to 0$; the theorem follows from (1).

REMARK. *Though a convex function necessarily possesses a derivative almost everywhere, not all sub-$F(x)$ functions possess this property.*

PROOF. Let $\phi(x)$ be a nowhere differentiable function; the family

$$F(x; \alpha, \beta) \equiv \phi(x) + \alpha x + \beta$$

satisfies conditions (1) and (2). Now any particular member of this family is itself a sub-$F(x)$ function, but is nowhere differentiable.

4. *Characterization of Sub-$F(x; \alpha, \beta)$ Functions.*

THEOREM 7. *A necessary and sufficient condition that a continuous function $f(x)$ be a sub-$F(x)$ function is that for all x_0, with $a<x_0<b$, and for all $\delta>0$, there exist a positive $h=h(x_0, \delta)<\delta$, with*

$$a < x_1 = x_0 - h < x_0 < x_0 + h = x_2 < b, \tag{32}$$

such that

$$f(x_0) \leqq F_{12}(x_0). \tag{33}$$

NECESSITY. If $f(x)$ is a sub-$F(x)$ function, then for all h such that (32) is satisfied, (33) holds as a special case of (3).

SUFFICIENCY. If $f(x)$ is not a sub-$F(x)$ function, then by (3) there exist x_3, x_4, x_5, with $a<x_3<x_5<x_4<b$, such that

$$f(x_5) > F_{34}(x_5). \tag{34}$$

By (34) and the continuity of $f(x)$ and of $F_{34}(x)$, there exist x_1, x_2, with $x_3 \leqq x_1 < x_5 < x_2 \leqq x_4$, such that

$$f(x) \geqq F_{34}(x), \qquad (x_1 \leqq x \leqq x_2), \tag{35}$$

the sign of equality holding at the end points but not elsewhere; then $F_{34}(x) \equiv F_{12}(x)$, so that (35) can be written as

$$f(x) > F_{12}(x), \qquad (x_1 < x < x_2). \tag{36}$$

Fix x_6, with $a<x_6<x_1$, and let $F_k(x)$ satisfy

$$F_k(x_6) = F_{12}(x_6), \qquad F_k(x_2) = F_{12}(x_2) + k = f(x_2) + k;$$

in particular,

$$F_0(x) \equiv F_{12}(x). \tag{37}$$

Let k increase continuously from $k=0$; then, for any $x>x_6$ in (a, b), by Theorems 1 and 2, $F_k(x)$ increases continuously. Further, it is easy to show, by Theorem 1 and the Heine-Borel Theorem, that for k sufficiently large,

$$f(x) < F_k(x), \qquad (x_1 \leqq x \leqq x_2). \tag{38}$$

By (36), (37), and (38), there is some largest $k_0>0$ for which

$$f(x) = F_{k_0}(x)$$

has a solution in $x_1 \leqq x \leqq x_2$. Let x_7 be the largest value for which

$$f(x_7) = F_{k_0}(x_7), \qquad (x_1 \leqq x_7 \leqq x_2). \tag{39}$$

Since $k_0>0$, we have $x_1<x_7<x_2$.

Suppose (33) could be satisfied at x_7 for the above function $f(x)$, which is not a sub-$F(x)$ function, and for some arbitrarily small positive h; we shall obtain a contradiction. Let h, with $x_1<x_8=x_7-h<x_{7+k}=x_9<x_2$, be a value for which (33) is satisfied:

$$f(x_7) \leqq F_{89}(x_7). \tag{40}$$

By the choice of k_0,

$$F_{89}(x_8) = f(x_8) \leqq F_{k_9}(x_8),$$
$$F_{89}(x_9) = f(x_9) < F_{k_9}(x_9),$$

so that, by Theorem 1 and its corollary,

$$F_{89}(x) < F_{k_9}(x), \qquad (x_8 < x < b);$$

in particular,

$$F_{89}(x_7) < F_{k_9}(x_7). \tag{41}$$

Now (41) contradicts (39) and (40).

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SUFFICIENT CONDITIONS FOR A NON-REGULAR PROBLEM IN THE CALCULUS OF VARIATIONS*

G. M. EWING

1. *Introduction.* Given $J = \int_{x_1}^{x_2} f(x, y, y')dx$, it is well known that a minimizing curve satisfies the necessary conditions of Euler, Weierstrass, and Legendre, which we shall designate as I, II, and III,† respectively. If further, $f_{y'y'}(x, y, y') \neq 0$ on the minimizing curve, the Jacobi condition IV is necessary, while the stronger set of conditions I, II$_b'$, III′, and IV′‡ are sufficient for a strong relative minimum.

The purpose of this study is to obtain a set of sufficient conditions for a curve without corners along which $f_{y'y'}$ may have zeros. Since the classical theory gives only the necessary conditions I, II, and III, we wish to obtain a Jacobi condition; and with this in view, introduce the integral

$$L \equiv \int_{x_1}^{x_2} \phi(x, y, y')dx, \ \phi(x, y, y') \equiv f(x, y, y') + k^2[y' - e'(x)]^2, \qquad (x_1 \leqq x \leqq x_2, k \lesseqgtr 0),$$

by means of which we find a necessary condition that we shall call IV$_L'$. Suitably strengthened, this becomes IV$_{Lb}'$ and the set of conditions I, II$_b$, III$_b$, and IV$_{Lb}'$ are found sufficient for an improper strong relative minimum.

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† G. A. Bliss, *Calculus of Variations*, 1925, pp. 130–132.

‡ Bliss, loc. cit., pp. 134–135.

It appears likely that analogous results can be obtained for other problems in the Calculus of Variations and I hope to discuss some of these at a later time.

2. *A Jacobi Necessary Condition.* If E: $y=e(x)$ furnishes at least an improper strong relative minimum for J, it furnishes a proper strong relative minimum for L. Furthermore, if E minimizes J it satisfies III for J. This implies that it satisfies III′ for L, since $\phi_{y'y'}=f_{y'y'}+2k^2$; and the classical treatment then shows that it must satisfy IV for L.

If E satisfies IV (or IV′) for every L, we shall say that it satisfies the condition IV_L (or IV'_L, respectively) for J. Clearly IV_L is necessary. We now show that the same is true of IV'_L.

We write the parameter in L in the form $k^2=(a^2+\alpha)/2$, $a\neq 0$, $\alpha>-a^2$, and consider the Jacobi differential equations*

$$qu'' + ru' + su = 0, \tag{1}$$

$$(q + a^2 + \alpha)u'' + ru' + su = 0, \tag{2}$$

for J and L, where $q=f_{y'y'}[x, e(x), e'(x)]\geqq 0$ in the closed interval $[x_1, x_2]$ from III, and r and s are other known functions of x. Since q may vanish in $[x_1, x_2]$, the usual existence theorems can not be applied to (1). They do apply to (2), however, the general solution of which for $\alpha=0$ is $u=c_1u_1(x)+c_2u_2(x)$, where the u's constitute a fundamental system and are of class C''† in $[x_1, x_2]$. $\Delta(x, x_1)=\pm u_2(x_1)u_1(x)\mp u_1(x_1)u_2(x)$ is a particular solution vanishing at $x=x_1$. By hypothesis, E: $y=e(x)$ is a minimizing curve satisfying IV_L so that, by proper choice of signs, $\Delta(x, x_1)$ is positive in the interval $x_1<x<x_2$.

For every admissible α (that is, $\alpha>-a^2$) there exists a solution $\Delta(x, x_1, \alpha)$ of (2) vanishing at $x=x_1$ and such that $\Delta'(x_1, x_1, \alpha)=\Delta'(x_1, x_1)$, where $\Delta''(x, x_1, \alpha)$ is continuous in x and of class C' in α.‡

We next study the related equation

$$(q + a^2)u'' + ru' + su = -\alpha\Delta''(x, x_1, \alpha), \tag{3}$$

* Oskar Bolza, *Vorlesungen über Variationsrechnung*, 1933, p. 60.

† That is, they have continuous second derivatives. Bolza, loc. cit., p. 14.

‡ Replace (2) by the system $u'=v$ and $(q+a^2+\alpha)v'+rv+su=0$, and apply the existence theorem given by Bolza, loc. cit., p. 187.

whose general solution can, by the method of variation of parameters, be expressed in the form

$$u = c_1 u_1(x) + c_2 u_2(x) + \alpha A(x, \alpha), \tag{4}$$

where

$$A(x, \alpha) = u_1(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha) u_2(x) dx}{(q + a^2) D(x)} - u_2(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha) u_1(x) dx}{(q + a^2) D(x)},$$

$$D(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \neq 0 \text{ in the closed interval } [x_1, x_2].*$$

$\Delta(x, x_1, \alpha)$, as a particular solution of (3), can be represented in the form (4); and, since it vanishes for $x = x_1$, we obtain

$$\Delta(x, x_1, \alpha) = \lambda \Delta(x, x_1) + \alpha A(x, \alpha), \tag{5}$$

where in general λ is a function of α. Clearly $\lambda(0) = 1$.

E satisfies IV_L by hypothesis. If it fails to satisfy IV′ for the L corresponding to $\alpha = 0$, we have

$$\Delta(x_2, x_1, 0) = \lambda(0) \Delta(x_2, x_1) = \Delta(x_2, x_1) = 0,$$

while, if a second $\alpha \neq 0$ has the same property, we have

$$\Delta(x_2, x_1, \alpha) = \lambda(\alpha) \Delta(x_2, x_1) + \alpha A(x_2, \alpha) = \alpha A(x_2, \alpha) = 0.$$

This requires

$$A(x_2, \alpha) = 0. \tag{6}$$

But

$$A(x_2, \alpha) = \int_{x_2}^{x_1} \frac{\Delta''(x, x_1, \alpha) \Delta(x, x_2)}{(q + a^2) D(x)} dx$$

$$= \frac{\Delta(\bar{x}, x_2)}{(\bar{q} + a^2) D(\bar{x})} \int_{x_2}^{x_1} \Delta''(x, x_1, \alpha) dx$$

$$= \frac{\Delta(\bar{x}, x_2) [\Delta'(x_1, x_1, \alpha) - \Delta'(x_2, x_1, \alpha)]}{(\bar{q} + a^2) D(\bar{x})},$$

$$(x_1 < \bar{x} < x_2; \bar{q} = q(\bar{x})),$$

* Bolza, loc. cit., p. 75.

where $\Delta(x, x_2)$ is written for $u_2(x_2)u_1(x) - u_1(x_2)u_2(x)$. This fraction can not vanish, the first factor in the numerator being different from zero by IV_L, the second factor being the difference between two terms of opposite sign. Thus (6) is false; and there is at most one L, namely the one for which $\alpha = 0$, for which E fails to satisfy IV′.

If $\Delta(x_2, x_1, 0) = 0$, we have $\Delta(x_2, x_1, \alpha) = \alpha A(x_2, \alpha)$ from (5). Furthermore $\Delta(x_2, x_1, \alpha)$ must then have a minimum of zero for $\alpha = 0$;* so that its derivative, which is $A(x_2, 0)$, must vanish. This is a special case of (6), which has been proved to be false, so that IV_L' is a necessary condition.

3. *Sufficient Conditions for a Minimum for L.* We assume an arc E: $y = e(x)$ satisfying the necessary conditions I, II, III, and IV_L' for J. If II is strengthened to II_b, we can show that this arc satisfies the classical sufficient conditions for L.

Comparing the Euler equations, we see that if E satisfies I for J it does the same for L. The E-functions† for the two problems are related by the equation

$$E_L(x, y, y', Y') \equiv E_J(x, y, y', Y') + k^2(y' - Y')^2,$$

so that II_b for J implies II_b' for L. We have seen in §2 that III for J implies III′ for L and the condition IV_L' requires IV′ for L as a matter of definition.

Hence E furnishes a proper strong minimum to L relative to a certain (x, y) region R, which in general depends on k.‡

4. *Sufficient Conditions for an Improper Strong Relative Minimum for J.* We must find how to strengthen our conditions so as to insure a field§ F which is independent of k. To that end we replace III by III_b and consider the line Λ: $x = x_1, y = n\lambda - y_1$, together with a slope function $P(\lambda) \equiv m\lambda + e'(x_1)$. The extremals for L are $y = y(x, a, b, \alpha)$, and the equations

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* $\Delta(x_2, x_1, \alpha) > 0$ for $\alpha \neq 0$ by IV_L and the choice of signs preceding equation (3).

† This is the only direct reference to the E-function. There need be no confusion with our notation for the curve E: $y = e(x)$.

‡ If E satisfies III′ and IV. but not IV′, for J, R reduces to the curve E as k approaches zero.

§ Bliss, loc. cit., pp. 132–33.

$$n\lambda + y_1 - y(x_1, a, b, \alpha) = 0,$$

$$m\lambda + e'(x_1) - y'(x_1, a, b, \alpha) = 0,$$

define $a = a(\lambda, \alpha) = \bar{a}(y, \alpha)$, and $b = b(\lambda, \alpha) = \bar{b}(y, \alpha)$* for any admissible α and for every y for which (x_1, y) is in the region where III_b holds. These implicit functions are of at least class C' in their respective variables. We thus have a family of extremals of parameter λ for each admissible α,

$$y = \phi(x, \lambda, \alpha) \equiv y[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha],$$

intersecting Λ and including E for $\lambda = 0$. We wish this family to furnish a field.

If there exists an x, $(x_1 < x \leqq x_2)$, such that $\phi(x, \lambda_1, \alpha) - \phi(x, \lambda_2, \alpha) = 0$, there is a $\bar{\lambda}$, $(\lambda_1 < \bar{\lambda} < \lambda_2)$, such that

$$\phi_\lambda(x, \bar{\lambda}, \alpha) = y_a \frac{\partial a}{\partial \lambda} + y_b \frac{\partial b}{\partial \lambda}\bigg|_{\lambda=\bar{\lambda}} = 0.$$

This can be expressed in the form†

$$\frac{n}{D_1}\begin{vmatrix} y_a(x) & y_b(x) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} - \frac{m}{D_1}\begin{vmatrix} y_a(x) & y_b(x) \\ y_a(x_1) & y_b(x_1) \end{vmatrix}, \tag{7}$$

$$D_1 = \begin{vmatrix} y_a(x_1) & y_b(x_1) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} \neq 0.\ddagger$$

We shall say that E satisfies the condition IV'_{Lb} if constants $\delta > 0$, $\eta > 0$, and A exist such that§

$$\Delta(x, x_1, y, \alpha) \equiv \begin{vmatrix} \bar{y}_a(x) & \bar{y}_b(x) \\ \bar{y}_a(x_1) & \bar{y}_b(x_1) \end{vmatrix}$$

is, in absolute value, greater than δ in the region $x_1 < x \leqq x_2$, $|y - y_1| \leqq \eta$, $A \geqq \alpha > -a^2$. The first determinant in (7) has a finite limit as n approaches zero; and hence, if n is small in absolute value, IV'_{Lb} insures that the expression will not vanish and that no two extremals of the family pass through the same

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* a, b, $\bar{a}$, and $\bar{b}$ also depend on m and n, which are omitted in the notation.

† $y_a(x)$, $\cdots$ are written for $y_a[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha]$, $\cdots$.

‡ The method used by Bolza, loc. cit., pp. 73–75, shows that $D_1 \neq 0$.

§ $\bar{y}_a(x)$, $\cdots$ are written for $y_a[x, \bar{a}(y, \alpha), \bar{b}(y, \alpha)\alpha]$, $\cdots$.

point. This condition also requires ϕ to be strictly monotone in λ for a given x and α, so that an extremal of the family passes through each point of a certain region F about E. The region F is a field and is independent of k (that is, of α).*

Finally, if E satisfies I, II_b, III_b, and IV'_{Lb}, we have $L(E) < L(C)$ for every $C \neq E$ in F. But

$$L(C) = J(C) + \epsilon, \qquad \epsilon > 0, \qquad \lim_{k=0} \epsilon = 0.$$

Furthermore $L(E) = J(E)$, so that $J(E) < J(C) + \epsilon$, and finally $J(E) \leqq J(C)$.

5. *Applications.* The line $y = 0$ is an extremal for a problem involving any one of the following integrands:

$$f \equiv M(x, y) + N(x, y)y', \qquad M_y = N_x,$$

$$f \equiv x^2 + y^2 + yy',\dagger$$

$$f \equiv y'^4.$$

Our sufficient conditions for an improper minimum are met by $y = 0$ in each case, but III′ is not met for any of them.

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* Condition IV'_{Lb} could be replaced by the following. There exist constants $\eta > 0$, $\xi > 0$, and A such that E satisfies III for $x_0 \leqq x \leqq x_2$, $x_0 = x_1 - \xi$ and such that $\Delta(x, x_0, y, \alpha) \neq 0$ for $x_0 = x_1 - \xi$, $x_0 < x \leqq x_2$, $|y - y_1| \leqq \eta$, $A \geqq \alpha > -a^2$. See Bolza, loc. cit., bottom p. 103.

† An example given by Bolza, loc. cit., p. 35.

A NOTE ON YOUNG-STIELTJES INTEGRALS*

BY F. G. DRESSEL

THEOREM 1. *If $f(x)$ is bounded and measurable Borel, and $g_1(x)$, $g_2(x)$ are of bounded variation, then the following equality holds:*

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x+0)dg_2(x) + \int_0^1 f(x)g_2(x-0)dg_1(x). \tag{1}$$

PROOF. In a recent article Evans† showed that if $g_1(x)$ and $g_2(x)$ have no common points of discontinuity, then

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x).$$

Therefore (1) holds if either $g_1(x)$ or $g_2(x)$ are continuous. It remains to show that the theorem holds when $g_1(x)$ and $g_2(x)$ are both step functions. Under these circumstances we have

$$\begin{aligned}
&\int_0^1 f(x)g_1(x+0)dg_2(x) + \int_0^1 f(x)g_2(x-0)dg_1(x) \\
&\quad = \sum f(\alpha_i)g_1(\alpha_i+0)[g_2(\alpha_i+0) - g_2(\alpha_i-0)] \\
&\qquad + \sum f(\alpha_i)g_2(\alpha_i-0)[g_1(\alpha_i+0) - g_1(\alpha_i-0)] \\
&\quad = \sum f(\alpha_i)[g_1(\alpha_i+0)g_2(\alpha_i+0) - g_1(\alpha_i-0)g_2(\alpha_i-0)] \\
&\quad = \int_0^1 f(x)d[g_1(x)g_2(x)],
\end{aligned}$$

where the summations are taken over all the discontinuities of $g_1(x)$ and $g_2(x)$.

The following lemmas are immediate applications of equation (1).

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* Presented to the Society, November 30, 1935.

† G. C. Evans, *Correction and addition to "Complements of potential theory,"* American Journal of Mathematics, vol. 57 (1935), pp. 623–626.

LEMMA 1. *If $f(x)$ is positive, bounded, and Borel measurable, and $g_1(x)$, $g_2(x)$ are monotone increasing, bounded, continuous on the left, then*

$$\int_0^1 f(x)d[g_1(x)g_2(x)] \leqq \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x). \tag{2}$$

LEMMA 2. *If in Lemma 1 the $g_1(x)$ and $g_2(x)$ are monotone decreasing functions continuous on the right, the inequality sign in (2) is reversed.*

W. C. Randels* used Lemma 1 in proving the existence of a solution of

$$f(x) = m(x) + \lambda \int_0^x f(y)dK(x, y).$$

In essentially the same manner, by making use of Lemma 2, we may prove the following theorem.

THEOREM 2. *If $g(x)$ is of bounded variation and if* (a) *$K(x, y)$ is Borel measurable in y for every x,* (b) *$K(x+0, y)=K(x, y)$,* (c) *$K(x, x)=0$,* (d) *$|K(x_1, y)-K(x_2, y)| \leqq |T(x_1)-T(x_2)|$, where the function $T(x)$ is bounded and non-decreasing with x, then*

$$f(x) = g(x) + \lambda \int_0^x K(x, y)df(y) \tag{3}$$

has a unique solution of bounded variation.

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* W. C. Randels, *On Volterra-Stieltjes integral equations*, Duke Mathematical Journal, vol. 1 (1935), pp. 538–542.

BRANCH-POINT MANIFOLDS ASSOCIATED WITH A LINEAR SYSTEM OF PRIMALS*

BY T. R. HOLLCROFT

1. *Introduction.* Linear ∞^α systems of primals in S_r have been treated† only for $\alpha=1, 2$. The properties of a linear system are obtained from the characteristics of the jacobian and of the branch-point manifold associated with the system. There are, at present, no means for deriving most of the characteristics of a singular primal or manifold in S_r, especially for $r>4$.

In this paper, a theorem is developed giving a set of characteristics of the branch-point manifolds of the system and its sub-systems. This is a step, not only toward the characterization of a general linear system in S_r, but also toward the study of singular manifolds which contain both nodal and cuspidal manifolds.‡

2. *Definitions and Basic Considerations.* In S_r, the linear ∞^r system, F_r, of primals is defined by the equation

$$\sum \lambda_i f_i = 0, \qquad (i = 1, 2, \cdots, r+1), \tag{1}$$

in which the f_i are general algebraic functions of order n in the $r+1$ homogeneous variables x_i. Then $f_i=0$ is the equation of a primal of order n without singularities in S_r.

The primals of F_r in the r-space (x) are in (1, 1) correspondence with the primes $\sum a_i y_i=0$, $(i=1, 2, \cdots, r+1)$, of an r-space (y). This correspondence is defined by the equations

$$\rho y_i = f_i, \qquad (i = 1, 2, \cdots, r+1).$$

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* Presented to the Society, September 12, 1935.

† T. R. Hollcroft, *Pencils of hypersurfaces*, American Journal of Mathematics, vol. 53 (1931), pp. 929–936; *Nets of manifolds in i dimensions*, Annali di Matematica, (4), vol. 5 (1927–28), pp. 261–267.

‡ These terms will be used: *node*, a double point of a manifold at which the quadric hypercone is entirely general; *nodal manifold of a manifold f*, a manifold for every point of which (except points on pinch and singular loci) the two tangent linear manifolds to f are distinct; *cuspidal manifold of f*, a manifold for all points of which the two tangent linear manifolds to f coincide; *cone* to mean *hypercone* for $r>3$.

To a point P of (y), considered as bearing ∞^{r-1} primes, corresponds n^r points of (x). These n^r points are the basis points of the ∞^{r-1} linear system of primals F_{r-1} in which the primals are in (1, 1) correspondence with the primes in (y) through P. Since (y) contains ∞^r points, F_r contains ∞^r linear systems F_{r-1}.

In the general case, to an S_k of (y), $(k \leqq r-1)$, considered as bearing ∞^{r-k-1} primes, corresponds in (x) the basis manifold M_k (of dimension k and order n^{r-k}) of an ∞^{r-k-1} linear system of primals F_{r-k-1} in which the primals are in (1, 1) correspondence with the primes of (y) through S_k. Since (y) contains $\infty^{(k+1)(r-k)}$ linear manifolds S_k, the system F_r contains $\infty^{(k+1)(r-k)}$ linear systems F_{r-k-1}.

The jacobian J of the linear system F_r is a primal of order $(r+1)(n-1)$. It is the locus of double points and contacts of primals of F_r. The jacobian J also contains the jacobian manifolds of all the linear systems of primals contained in F_r such that the jacobians of the systems F_{r-k-1} form a $(k+1)(r-k)$-parameter linear system of manifolds on J. Likewise J contains the singularities of higher order and contacts of higher order of primals of F_r and of all linear systems of primals contained in F_r. The jacobian J has no actual singularities, only apparent singular manifolds.

The (1, 1) correspondence between the primals of F_r and the primes of (y) establishes a $(1, n^r)$ involution between the points of (y) and (x), and J is the locus of coincidences of this involution. The image of J in (y) is the branch-point primal L, the locus of points such that all primals of each associated F_{r-1} have contact with a line at a point on J. The ∞^{r-1} contacts generate J.

L is also the envelope of primes of (y) which correspond to primals of F_r that have a node. To the points of contact of primes with L correspond uniquely the nodes, which lie on J.

The order μ_0 of L is the number of points in which J and $r-1$ primals of F_r intersect, that is, $\mu_0=(r+1)(n-1)n^{r-1}$.

The classes of L are defined as follows:

μ_1, the order of the tangent cone to L from a point;
μ_2, the order of the tangent cone to L from a line;

. . . . . .

μ_{k+1}, the order of the tangent cone to L from an S_k;

. . . . . .

μ_{r-1}, the number of tangent primes to L from an S_{r-2}.

3. *A Theorem Defining Branch-Point Manifolds of the F_{r-k-1}.* The primals of an F_{r-k-1} of F_r of (x) are in (1, 1) correspondence with the primes of S_{r-k-1}, a sub-space of (y). This establishes a (1, 1) correspondence between the points of S_{r-k-1} and the basis manifolds M_{k-1} of the $(r-k-2)$-parameter linear systems of primals contained in F_{r-k-1}. The locus of points of S_{r-k-1} for which all of the primals of the associated $(r-k-2)$-parameter linear systems have contact at one point with a line is the branch-point manifold L_{r-k-2} (primal of S_{r-k-1}) and the locus of contacts in (x) is the jacobian manifold J_{r-k-2}.

As shown in §2, the primals of an $(r-k-1)$-parameter linear system of primals belonging to F_r in (x) are in (1, 1) correspondence with the primes of (y) through an S_k. The $(k+1)$st class of L, μ_{k+1}, is the order of the tangent cone enveloped by primes through S_k tangent to L. To each such tangent prime corresponds a primal of F_{r-k-1} and of F_r with a node.

Consider any given S_{r-k-1} of (y). S_{r-k-1} intersects each of the primes through S_k in an S_{r-k-2}, which is a prime of S_{r-k-1}. The primals of F_{r-k-1} are in (1, 1) correspondence with these primes $[S_{r-k-2}$ of $(y)]$ of S_{r-k-1}.

Since the order of the tangent cone to L from S_k is μ_{k+1}, the section of this tangent cone by S_{r-k-1} is a manifold V_{r-k-2} of dimension $r-k-2$ and order μ_{k+1}. This manifold V_{r-k-2} is the envelope of the primes of S_{r-k-1} which are sections by S_{r-k-1} of the primes of (y) through S_k tangent to L. Therefore the primes in S_{r-k-1} enveloping V_{r-k-2} are in (1, 1) correspondence with the primals of F_{r-k-1} which have a node. But, as previously shown, the (1, 1) correspondence between the primals of F_{r-k-1} and the primes of S_{r-k-1} establish an involution in which the branch-point manifold L_{r-k-2} of S_{r-k-1} is defined as the envelope of primes of S_{r-k-1} which correspond uniquely to primals of F_{r-k-1} that have a node. Therefore, in S_{r-k-1},

$$L_{r-k-2} \equiv V_{r-k-2}.$$

This identity establishes the following theorem.*

The section by an S_{r-k-1} of the tangent cone from an S_k to L, where L is the branch-point primal in the r-space (y) associated

* This theorem has been established for three dimensions. See T. R. Hollcroft, *The general web of algebraic surfaces of order n and the involution defined by it*, Transactions of this Society, vol. 35 (1933), p. 859.

with an r-parameter linear system of primals F_r of an r-space (x), is the branch-point manifold L_{r-k-2} of S_{r-k-1} associated with a linear $(r-k-1)$-parameter system of primals F_{r-k-1} belonging to F_r.

The order μ_{k+1} of L_{r-k-2} is also the order of the contour manifolds on L of the tangent cones from an S_k. These contour manifolds, of dimension $r-k-2$, form a linear system on L and are the respective images of the jacobian manifolds of the F_{r-k-1} contained in F_r. These jacobian manifolds form a linear system on J of the same respective dimension as the associated linear system of contour manifolds on L. Its contour manifold, L_{r-k-2}, and its associated jacobian manifold are all in (1,1) correspondence.

4. *Relations Resulting from the Theorem.* By the above theorem, the $(k+1)$st class μ_{k+1} of L is the order of the branch-point manifold L_{r-k-2} associated with an F_{r-k-1} belonging to F_r.

In the $(1, n^{r-k-1})$ involution associated with F_{r-k-1}, the condition for a point to lie on L_{r-k-2} is that the primals of F_{r-k-1} have a common tangent S_{k+2} at a common point. The condition that $r-k-1$ primals have a common tangent S_{k+2} at a common point is the tact-invariant of this system of primals. The order of this tact-invariant is*

$$\mu_{k+1}=\frac{1}{(k+2)!}(r+1)r(r-1)(r-2)\cdots(r-k)(n-1)^{k+2}n^{r-k-2}.$$

This is, therefore, the order of L_{r-k-2} and the value of μ_{k+1}, the $(k+1)$st class of L.

The order μ_0 of L results from the above formula for $k=-1$, that is, the order μ_0 is the tact-invariant of r primals of F_r. The final class of L, $\mu_{r-1}=(r+1)(n-1)^r$, is the order of the discriminant of a primal of F_r and is not a tact-invariant, since it involves only one primal. The value of μ_{r-1}, however, is also given by the above formula for $k=r-2$.

The class μ_{r-2} of L is the order of the tangent cone to L from an S_{r-3}. This is also the order of the branch-point curve L_1 associated with a net of primals of F_r. The complete set of charac-

* T. R. Hollcroft, *Tact-invariants of primals in S_r*, Journal of the London Mathematical Society, vol. 11 (1936), p. 24.

teristics of L_1 is given in a former paper.* These are also the characteristics of a tangent cone (surface) to L from an S_{r-3}. The characteristics of L_2 and therefore of the tangent cone to L from an S_{r-4} have been found† for $n=2$, but not for a general n.

Since the final class‡ of a section of L made by an S_{k+2} is μ_{k+1}, the above value of μ_{k+1} gives the final classes of all sections of L by a linear manifold as well as the orders of all tangent cones to L from a linear manifold. The order of the section of L by any linear manifold is μ_0.

In general, L in (y) has both a nodal and a cuspidal manifold, each of dimension $r-2$, and these manifolds are themselves singular. For a linear system of dimension r in S_{r-1}, however, L has only a nodal manifold of dimension $r-2$, containing a pinch manifold of dimension $r-3$.

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* T. R. Hollcroft, *Nets of manifolds in i dimensions*, loc. cit.

† T. R. Hollcroft, *The web of quadric hypersurfaces in r dimensions*, this Bulletin, vol. 41 (1935), pp. 97–103.

‡ By *final class* of an S_{k+2} section of L is meant the number of S_{k-1} through an arbitrary S_k (all in S_{k+2}) tangent to L.

NOTE ON THE IDEALS OF CYCLIC ALGEBRAS*

BY RALPH HULL

1. *Introduction.* The purpose of this note is the generalization of certain results in a recent paper by Latimer† on the number of ideals of given norm in a generalized quaternion algebra.

We consider rational cyclic division algebras D of prime degree $n(\geqq 2)$ over the field R of rational numbers. Let $\mathfrak{o}$ be any maximal order‡ of D. The reduced discriminant of $\mathfrak{o}$ is an invariant $\Delta=\Delta(D)$ of D called the discriminant of D, and is of the form $\Delta=\pm\sigma^{n(n-1)}$, where $\sigma=q_1\cdots q_s$ is the product of the distinct rational primes $q_1\cdots q_s$ which are ramified§ in D. For each such q, the two-sided ideal $q\mathfrak{o}$ is the nth power of an indecomposable two-sided prime ideal of $\mathfrak{o}$, and the q-adic extension D_q is a division algebra of degree n of R_q. For all other rational primes p, D_p is the algebra of all matrices of degree n over R_p and $\mathfrak{o}p$ is a two-sided prime ideal of $\mathfrak{o}$ having only one-sided ideal divisors.

By a (normal) ideal of D is meant an ideal (one or two-sided) with respect to some maximal order of D. Such an ideal is called integral if it is contained in its right or left order. We denote various maximal orders by $\mathfrak{o}$, $\mathfrak{o}_1$, $\mathfrak{o}_2$, $\cdots$, and an ideal $\mathfrak{a}$ by $\mathfrak{a}_{ij}$ if $\mathfrak{o}_i\mathfrak{a}=\mathfrak{a}\mathfrak{o}_j=\mathfrak{a}$ and it is necessary to indicate $\mathfrak{o}_i$ and $\mathfrak{o}_j$. The (reduced) norm of an ideal is an ideal of R such that, for a principal ideal $\alpha\mathfrak{o}$ (or $\mathfrak{o}\alpha$), α in D, $N(\alpha\mathfrak{o})$ (or $N(\mathfrak{o}\alpha)$) $=(N(\alpha))$, where $N(\alpha)$ is the reduced norm corresponding to the rank equation of degree n. Our object is to prove the following result.

THEOREM. *Let $\mathfrak{o}$ be any maximal order of D and let A be any rational integer. Write $A=A_1A_0$, where A_0 is prime to $\Delta(D)$ and every prime factor of A_1 divides $\Delta(D)$. Then the number of integral*

* Presented to the Society, April 9, 1937.

† C. G. Latimer, Transactions of this Society, vol. 40 (1936), pp. 439–450.

‡ Maximal orders for all D have been given explicitly. See Albert, this Bulletin, vol. 40 (1934), pp. 164–176, for $n=2$, and Hull, Transactions of this Society, vol. 38 (1935), pp. 515–530, for $n>2$.

§ We refer to Deuring, *Algebren*, Ergebnisse der Mathematik, Chapter VI, for all definitions and theorems used here except when explicit reference elsewhere is given.

$\mathfrak{o}$-right ideals of norm (A) is equal to the number of classes of right associated integral matrices of degree n and determinant A_0.

Two integral matrices M_1 and M_2 are said to be right associated* if there is an integral matrix U such that $|U| = \pm 1$ and $M_2 = M_1 U$. In case $n = 2$, the number of such classes of matrices of given determinant A_0 is easily seen to be the sum of the divisors† of A_0.

2. *Preliminary Lemmas.* We need two lemmas easily obtained from the general ideal theory of linear algebras (Deuring, loc. cit.).

Let $\mathfrak{a} = \mathfrak{a}_{r+1,1}$ be an integral ideal and let

$$N(a) = (A), \qquad A = p_r^{\gamma_r} \cdots p_1^{\gamma_1}, \tag{1}$$

where the p's are distinct rational primes.

LEMMA 1. *The ideal $\mathfrak{a} = \mathfrak{a}_{r+1,1}$ has a special factorization*

$$\mathfrak{a} = \mathfrak{a}_{r+1,r}^{(r)} \cdot \mathfrak{a}_{r,r-1}^{(r-1)} \cdots \mathfrak{a}_{2,1}^{(1)}, \quad \mathfrak{a}^{(i)} \text{ integral}, \quad N(\mathfrak{a}^{(i)}) = p_i^{\gamma_i}. \tag{2}$$

For a given order $p_r, \cdots, p_1$ of the distinct prime divisors of A in (1), the special factorization (2) is unique.

The existence of (2) is implied by the fact that there exists a factorization of $\mathfrak{a}$ into indecomposable ideals in which the order of the prime ideals to which the factors belong is arbitrarily assigned (Deuring, p. 106). To prove the uniqueness claimed we consider p-components and apply a theory due to Hasse (Deuring, pp. 94–107).

Let p be any rational prime. Then from (2)

$$(\mathfrak{a})_p = (\mathfrak{a}^{(r)})_p \cdots (\mathfrak{a}^{(1)})_p,$$

where $(\mathfrak{a})_p$ denotes the p-component, that is, p-adic limit set of $\mathfrak{a}$. If $(p, A) = 1$, $(\mathfrak{a})_p$ is a maximal order of D_p, and since $\mathfrak{o}_{r+1}\mathfrak{a} = \mathfrak{a}\mathfrak{o}_1 = \mathfrak{a}$, we have $(\mathfrak{a})_p = (\mathfrak{o}_{r+1})_p = (\mathfrak{o}_1)_p$. Similarly $(\mathfrak{a}^{(1)})_p = (\mathfrak{o}_2)_p = (\mathfrak{o}_1)_p$, $(\mathfrak{a}^{(2)})_p = (\mathfrak{o}_3)_p = (\mathfrak{o}_2)_p = (\mathfrak{o}_1)_p$, and so on, and it is clear that $(\mathfrak{a}^{(i)})_p = (\mathfrak{a})_p$, $(i = 1, \cdots, r)$. If $p = p_i$, $(i = 1, \cdots, r)$,

* MacDuffee, *The Theory of Matrices*, Ergebnisse der Mathematik, Chapter III.

† See Latimer, loc. cit.

we find in a similar way that $(\mathfrak{a}^{(r)})_p = \cdots = (\mathfrak{a}^{(i+1)})_p = (\mathfrak{o}_{r+1})_p$ and $(\mathfrak{a}^{(i-1)})_p = \cdots = (\mathfrak{a}^{(1)})_p = (\mathfrak{o}_1)_p$, whence $(\mathfrak{a}^{(i)})_p = (\mathfrak{a})_p$. Hence, for every rational prime p, the p-component of each $\mathfrak{a}^{(i)}$ is uniquely determined by $\mathfrak{a}$. It follows that each $\mathfrak{a}^{(i)}$ is uniquely determined by $\mathfrak{a}$ since each is determined by the totality of its p-components.

LEMMA 2. *If p is ramified in D, there is exactly one ideal of D_p of given norm p^ν. If p is not ramified in D, the number of right ideals of D_p with respect to a given maximal order of D_p, of given norm p^ν, is the number $\psi(p^\nu)$ of classes of right associated rational integral matrices of degree n and determinant p^ν.*

For the first part of Lemma 2 we have only to note that every ideal of D_p, p ramified in D, is a power of the single prime ideal of the unique maximal order of D_p. The second part is seen as follows. A maximal order $\mathfrak{o}_p$ of D_p, p not ramified in D, is of the form $\mathfrak{o}_p = \sum c_{ij}\mathfrak{g}_p$, where the c_{ij}, $(i, j = 1, \cdots, n)$, are matrix units and $\mathfrak{g}_p$ is the maximal order of R_p. Every $\mathfrak{o}_p$-right ideal is a principal ideal $\alpha\mathfrak{o}_p$, where α is of the form (Deuring, loc. cit. p. 101)

$$(3) \quad \alpha = p^{\mu_1}c_{11} + d_{21}c_{21} + p^{\mu_2}c_{22} + \cdots + d_{n1}c_{n1} + \cdots + p^{\mu_n}c_{nn},$$

where $\mu_1 + \mu_2 + \cdots + \mu_n = \nu$, and d_{ij} is uniquely determined modulo p^{μ_j}. The last part of the lemma is obvious from (3).

3. *Proof of the Theorem.* Since every integral ideal $\mathfrak{a}$ of norm (A), A as in (1), has the unique special factorization (2), we proceed to count the number of possible distinct sets $\mathfrak{a}^{(1)}, \cdots, \mathfrak{a}^{(r)}$ which yield an $\mathfrak{a}$.

Consider first $\mathfrak{a}^{(1)}$ whose right order is the fixed maximal order $\mathfrak{o}_1 = \mathfrak{o}$, of the theorem. For every rational prime $p \neq p_1$, we have $(\mathfrak{a}^{(1)})_p = (\mathfrak{o})_p$, and for $p = p_1$, $(\mathfrak{a}^{(1)})_p$ is a right ideal with respect to $(\mathfrak{o})_p$ of norm $p_1^{\nu_1}$. Thus $(\mathfrak{a}^{(1)})_p$ is unique for all $p \neq p_1$ and, by Lemma 2, there are precisely 1 or $\psi(p_1^{\nu_1})$ possibilities for $(\mathfrak{a}^{(1)})_{p_1}$ according as p_1 is ramified or unramified in D. Hence by an argument used in the proof of Lemma 1, there are precisely 1 or $\psi(p_1^{\nu_1})$ possibilities for the factor $\mathfrak{a}^{(1)}$ in the respective cases.

Suppose, next, that $\mathfrak{a}^{(1)}$ is fixed and consider $\mathfrak{a}^{(2)}$, whose right order $\mathfrak{o}_2$ is uniquely determined by $\mathfrak{a}^{(1)}$, since $\mathfrak{o}_2$ is the left order of $\mathfrak{a}^{(1)}$. The same argument made for $\mathfrak{a}^{(1)}$ and $\mathfrak{o}_1$ applies to $\mathfrak{a}^{(2)}$ and $\mathfrak{o}_2$, and we can proceed similarly with $\mathfrak{a}^{(3)}, \mathfrak{a}^{(4)}, \cdots$, suc-

cessively. It is plain that the total number of sets $\mathfrak{a}^{(1)}, \cdots, \mathfrak{a}^{(r)}$ is $\prod\psi(p_j^{ri})$, where j ranges over those of the integers $1, \cdots, r$ for which p_j is unramified in D, that is $\prod(p_j^{ri}) = A_0$. To complete the proof of the theorem we have to show that $\prod\psi(p_j^{ri}) = \psi(A_0)$. This follows from the following lemma.

LEMMA 3. *If $A = BC$, where A, B, C are rational integers such that $(B, C) = 1$, then $\psi(A) = \psi(B)\psi(C)$.*

To prove this lemma we apply the methods of §2 to the *simple* algebra S of all rational matrices of degree n. For a given system of matrix units c_{ij}, $(i, j = 1, \cdots, n)$, the set $\mathfrak{m} = \sum c_{ij}\mathfrak{g}$, where $\mathfrak{g}$ denotes the set of rational integers, is a maximal order of S. Every integral $\mathfrak{m}$-right ideal is a principal ideal (MacDuffee, loc. cit.) $\mathfrak{a} = M_1\mathfrak{m}$, where M_1 is an integral matrix and the reduced norm of $\mathfrak{a}$ is $(|M_1|) = (A)$, say. If also $\mathfrak{a} = M_2\mathfrak{m}$, then $M_2 = M_1U$, U integral, $|U| = \pm 1$. Hence $\psi(A)$ is the number of integral $\mathfrak{m}$-right ideals of norm (A). In S, we have unique special factorizations* similar to those of Lemma 1. Hence if $A = BC$, we can count the number of $\mathfrak{m}$-right ideals of norm (A) by counting the number of right ideals, with respect to certain maximal orders of S, of norms (B) and (C). This yields the lemma.

4. *Applications of the Theorem.* In case the class number h of D is one,† our theorem yields interesting results concerning representations‡ by the norm form associated with a maximal order $\mathfrak{o}$ of D. Thus, if $h = 1$, every ideal of D is principal, $\mathfrak{a}_{ij} = \alpha\mathfrak{o}_j = \mathfrak{o}_i\beta$, α, β in D. If also, $a_{ij} = \alpha'\mathfrak{o}_j$, then $\alpha' = \alpha u$, where u is a unit of $\mathfrak{o}_j$, and the norm form associated with $\mathfrak{o}_j$ is universal. The number of sets of integral representations§ of A is $\psi(A_0)$. The representations of a set are connected by the automorphs of the form associated with the units of $\mathfrak{o}$.

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* Every rational prime is unramified in S.

† M. Eichler, Journal für Mathematik, vol. 176 (1937), pp. 192–202, has proved general results on the class number of algebras which imply $h = 1$ for all D with $n > 2$ and for rational quaternion algebras with real splitting fields.

‡ Cf. L. E. Dickson, *Algebren und ihre Zahlentheorie*, §100.

§ We must have $A > 0$ for $n = 2$, D definite. In this case, the infinite prime spot of R is said to be ramified in D.

POLYNOMIAL APPROXIMATION ON A CURVE OF THE FOURTH DEGREE*

BY DUNHAM JACKSON

1. *Introduction.* In the study of orthogonal polynomials and polynomial approximation, or of the corresponding theory for trigonometric sums, a powerful auxiliary is Bernstein's theorem on the derivative of a trigonometric sum or of a polynomial. When it is desired to investigate similar problems relating to approximation by means of polynomials on a curve in the plane of two real variables† x and y, the question arises whether something in the nature of Bernstein's theorem is available in this case also. For some of the simplest curves, such as a line segment or a circle, the question is merely one of interpretation, the polynomials in x and y reducing at once to polynomials or trigonometric sums in terms of the arc length (or a constant multiple of it) as parameter. For any curve segment of the form $x=\phi(t)$, $y=\psi(t)$, where $\phi(t)$ and $\psi(t)$ are polynomials in t, there is an immediate answer as far as differentiation with respect to t is concerned; and if $|\phi'(t)|+|\psi'(t)|$ is everywhere positive (and so has a positive lower bound) on the closed range of values of t considered, a derivative with respect to arc length does not exceed a constant multiple of the derivative with respect to t. This paper is concerned with an illustrative case in which the problem appears to be not entirely trivial, and yet susceptible of simple and elementary treatment. An appropriate form of Bernstein's theorem is obtained, and its application to a problem of polynomial approximation is indicated.

A concluding paragraph relates to the convergence in the mean of developments in series of orthogonal polynomials on an arbitrary curve.

2. *Bernstein's Theorem.* Let C be the curve

$$x^4 + y^4 = 1.$$

* Presented to the Society, December 31, 1936.

† See D. Jackson, *Orthogonal polynomials on a plane curve*, presently to appear in the Duke Mathematical Journal.

It can be represented parametrically by the equations

$$x = |\cos\theta|^{1/2} \operatorname{sgn}\cos\theta, \qquad y = |\sin\theta|^{1/2} \operatorname{sgn}\sin\theta.$$

Let s be the arc length, measured counter-clockwise from the point $(1, 0)$. Let $P(x, y)$ be a polynomial of the nth degree in the variables x and y together, and let

$$|P(x, y)| \leqq L$$

at all points of the curve. *It is to be shown that*

$$\left|\frac{d}{ds}P(x, y)\right| \leqq AnL$$

everywhere on the curve, A being an absolute constant.

Let

$$\alpha_1(x, y) = \frac{1}{4}[P(x, y)+P(-x, y)+P(x, -y)+P(-x, -y)],$$

$$\beta_1(x, y) = \frac{1}{4}[P(x, y)-P(-x, y)+P(x, -y)-P(-x, -y)],$$

$$\gamma_1(x, y) = \frac{1}{4}[P(x, y)+P(-x, y)-P(x, -y)-P(-x, -y)],$$

$$\delta_1(x, y) = \frac{1}{4}[P(x, y)-P(-x, y)-P(x, -y)+P(-x, -y)].$$

Each of these polynomials has L as an upper bound for its absolute value on C. They are respectively of the form

$$\alpha_1(x, y) = \alpha_2(x^2, y^2), \qquad \beta_1(x, y) = x\beta_2(x^2, y^2),$$
$$\gamma_1(x, y) = y\gamma_2(x^2, y^2), \qquad \delta_1(x, y) = xy\delta_2(x^2, y^2),$$

where α_2, β_2, γ_2, δ_2 are polynomials of degree not exceeding $n/2$ and so (for $n \geqq 1$) not exceeding $n-1$, in the two arguments together.

Since a polynomial in x^2 and y^2 is a polynomial in $|\cos\theta|$ and $|\sin\theta|$ and consequently, for $0 \leqq \theta \leqq \pi/2$, a trigonometric sum in θ, the expressions $\alpha_1, \cdots, \delta_1$ can be written as

$$\alpha_1(x, y) = \alpha(\theta), \qquad \beta_1(x, y) = (\cos\theta)^{1/2}\beta(\theta),$$
$$\gamma_1(x, y) = (\sin\theta)^{1/2}\gamma(\theta), \qquad \delta_1(x, y) = (\sin\theta\cos\theta)^{1/2}\delta(\theta)$$

for $0 \leqq \theta \leqq \pi/2$, where $\alpha(\theta), \cdots, \delta(\theta)$ are trigonometric sums of order not greater (and in general considerably less) than $n-1$. From extensions of Bernstein's theorem established elsewhere* it follows that the absolute value of the derivative of each of the last four right-hand members with respect to θ has for $0<\theta<\pi/2$ an upper bound of the form

$$\frac{A_1 nL}{(\sin\theta\cos\theta)^{1/2}},$$

where A_1 is an absolute constant; it is to be noted that as regards the orders of magnitude concerned $(\sin\theta)^{1/2}$, $(\cos\theta)^{1/2}$ are equivalent to $\theta^{1/2}$, $(\pi/2-\theta)^{1/2}$.

On passage to the second, third, or fourth quadrant the trigonometric sums $\alpha, \cdots, \delta$ are modified by substitution of $-\cos\theta$ for $\cos\theta$, or of $-\sin\theta$ for $\sin\theta$, or both, in the polynomial expressions, but the form of the conclusion is unchanged except for the replacement of $\sin\theta$ and $\cos\theta$ by their absolute values. Since $P(x, y) = \alpha_1+\beta_1+\gamma_1+\delta_1$, there is an absolute constant A_0 such that

$$\left| \frac{d}{d\theta} P(x, y) \right| \leqq \frac{A_0 nL}{|\sin\theta\cos\theta|^{1/2}}$$

in all four quadrants.

For $0<\theta\leqq\pi/4$,

$$\frac{ds}{d\theta} \geqq \frac{dy}{d\theta} = \frac{\cos\theta}{2(\sin\theta)^{1/2}} \geqq \frac{2^{-3/2}}{(\sin\theta)^{1/2}} \geqq \frac{2^{-7/4}}{(\sin\theta\cos\theta)^{1/2}},$$

and the relation $ds/d\theta \geqq |dx/d\theta|$ leads to the same result for $\pi/4 \leqq \theta < \pi/2$. So

$$\frac{d\theta}{ds} \leqq 4(\sin\theta\cos\theta)^{1/2}$$

throughout the interval $(0, \pi/2)$. The same conclusion, with $|\sin\theta\cos\theta|^{1/2}$ in the right-hand member, holds in the other quadrants also. It follows that

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* See D. Jackson, *Bernstein's theorem and trigonometric approximation*, Transactions of this Society, vol. 40 (1936), pp. 225–251; Theorems 1, 3, 4.

$$\left| \frac{d}{ds} P(x, y) \right| \leqq 4A_0 nL$$

everywhere on the curve, in agreement with the assertion that was to be proved.*

3. *Convergence.* This proposition can be used in connection with the theory of closest polynomial approximation on the curve C just as the standard form of Bernstein's theorem is used in the theory of approximation by trigonometric sums.†

For detailed interpretation of the results information is needed as to the possibility of representing a given function of s on C by means of polynomials in x and y with a specified degree of accuracy. The discussion here will be restricted to a simple case illustrating the accessibility of the question to treatment.

Let s be measured counter-clockwise from the point $(1, 0)$, as already suggested, let σ be the total length of the curve, and *let $f(s)$ be a continuous function of period σ satisfying a Lipschitz condition with respect to s.* Let a function $F(x, y)$ be defined throughout the (x, y)-plane by the specifications that it shall vanish at the origin, shall be linear on each ray issuing from the origin, and on each of these rays shall coincide in value with $f(s)$ at the point where the ray meets the curve. Then $F(x, y)$ satisfies a Lipschitz condition as a function of the two variables.

(This observation, evident by geometric intuition applied to the surface $z = F(x, y)$, may be analyzed as follows. If O is the origin, and if P and Q are two points lying on a curve $x^4+y^4 =$constant, subject to the restriction that the magnitude of the angle POQ does not exceed $\pi/2$, say, the ratio of the arc PQ to the chord PQ does not exceed a constant. Also, the angle PQO has a positive lower bound and an upper bound less than π. Let P and R be any two points of the plane for which the magnitude of the angle POR is not greater than $\pi/2$. Let Q be the point in which the ray OR meets that curve $x^4+y^4 =$constant which

* Although the proof does not apply in the first instance at the quadrantal points where $\sin\theta\cos\theta = 0$, the validity of the conclusion at these points follows by continuity.

† See, for example, D. Jackson, *Certain problems of closest approximation*, this Bulletin, vol. 39 (1933), pp. 889–906; Lemmas 1 and 5, Theorems 1, 2, 9, and 10.

passes through P. In the (rectilinear) triangle PQR, we have $PQ:PR=\sin R:\sin Q$, $QR:PR=\sin P:\sin Q$; as $\sin Q=\sin PQO$ has a positive lower bound, each of the distances PQ, QR is less than a constant multiple of PR. The ratio of $|F(P)-F(Q)|$ to the length of the arc PQ is the same as for the corresponding points on the given curve $x^4+y^4=1$, and so does not exceed the constant of the Lipschitz condition of the hypothesis. Hence $|F(P)-F(Q)|$ is not greater than a constant multiple of the chord PQ, and not greater than a constant multiple of PR. On the ray OR, where F is a linear function of distance with coefficient less than or equal to the corresponding value of $|f(s)|$, $|F(Q)-F(R)|$ does not exceed a constant multiple of QR, because of the boundedness of $f(s)$, and so is not greater than a constant multiple of PR. Therefore $|F(P)-F(R)|$ does not exceed a constant multiple of PR. If the angle POR is greater than $\pi/2$, $|F(P)-F(O)|$ does not exceed a constant multiple of OP, $|F(O)-F(R)|$ does not exceed a constant multiple of OR, each of the distances OP, OR is less than PR, and $|F(P)-F(R)|$ again does not exceed a constant multiple of the distance PR.)

Consequently* there exist polynomials $P_n(x, y)$, of the nth degree in x and y, approximating $F(x, y)$ throughout the square $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, say, with maximum error not exceeding a constant multiple of $1/n$. In particular $f(s)$, to which $F(x, y)$ reduces on C, is represented by polynomials of the nth degree in x and y with the same upper bound for the magnitude of the error.

Taken in conjunction with the theorem of the first part of the paper, by the use of a type of argument referred to above as employed elsewhere in the theory of approximation by trigonometric sums, this means that under the hypothesis that has been imposed on $f(s)$ *if polynomials $P_n(x, y)$ of degree n are determined for successive values of n so as to minimize the integral*

$$\int_C |f(s) - P_n(x, y)|^m ds,$$

* See, for example, D. Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen . . .*, Dissertation, Gottingen, 1911, pp. 16–17, 88; E. L. Mickelson, *On the approximate representation of a function of two variables*, Transactions of this Society, vol. 33 (1931), pp. 759–781, §5.

with $m>1$, the polynomials $P_n(x, y)$ will converge uniformly on C toward $f(s)$. In particular, for $m=2$, *the expansion of $f(s)$ in series of polynomials orthogonal on C, with parameter s and weight function* 1, *will converge uniformly toward $f(s)$.* The statements can be generalized immediately by introduction of a suitably restricted weight function in the integral to be minimized.

4. *Convergence in the Mean.* As regards the convergence of developments in series of orthogonal polynomials, the proof just presented suggests an approach to the problem of convergence in the mean, not merely for the special curve which has been under consideration, but for an arbitrary curve, with an arbitrary parametric representation. Let a curve C (algebraic or non-algebraic) be given by a pair of equations $x=\phi(t), y=\psi(t)$, where ϕ and ψ are continuous for $0 \leqq t \leqq a$ (or continuous everywhere and of period a), and no two distinct values of t (or no two distinct values in a period) correspond to a single point (x, y), and let a corresponding system of orthogonal polynomials be constructed as described in the paper to which reference is made in the footnote to the opening paragraph above, the weight function, for simplicity of statement, being taken as 1. If $f(t)$ is any function continuous over the range of t, a function $F(x, y)$ can be formed* which is continuous in the two variables throughout a rectangle containing C, and reduces on C to $f(t)$. This $F(x, y)$ can be uniformly approximated throughout the rectangle by polynomials in x and y with any degree of accuracy, and on the curve C in particular $f(t)$ is uniformly approximated, and so approximated in the mean, by the same polynomials. It follows that any function $g(t)$ which is of class L^2 for $0 \leqq t \leqq a$ can be approximated in the mean with index 2 by polynomials in x and y, and hence by the least-square property that *the development of $g(t)$ in series of orthogonal polynomials converges in the mean* with this index.

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* See Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63–89, p. 63; also, for example, D. Jackson, *An elementary boundary value problem*, this Bulletin, vol. 22 (1915–16), pp. 393–397.

GENERAL TENSOR ANALYSIS*

BY A. D. MICHAL

1. *Introduction.* The *coordinates* of the traditional tensor analysis are points in a finite dimensional arithmetic space while those of the author's infinitely dimensional tensor analysis are points (functions) in a function space. The purpose of the present paper is to present the elements of a general tensor analysis that will include as instances the tensor calculi just mentioned. This general point of view has already been instrumental in the more elegant and further development of the function space instances.

For a space of *coordinates* we take a Banach space. The geometrical objects studied are contravariant vector fields, linear connections, multilinear forms and their covariant differentials. Non-holonomic geometric objects are not considered here, as I intend to pursue their study elsewhere.

2. *Abstract Coordinate Transformations.* We consider a Hausdorff† topological space T whose neighborhoods are mapped homeomorphically on an open set S of a Banach space E by mapping functions called *coordinate systems.*‡ If two neighborhoods intersect we have two mappings of their intersection on open subsets S_1 and S_2 of S. This establishes a homeomorphism $\bar{x}(x)$, called a *coordinate transformation*, that takes an open set $S_1 \subset S$ into an open set $S_2 \subset S$. As further restrictions we demand that the Fréchet differentials $\bar{x}(x; \delta x)$ and $x(\bar{x}; \delta\bar{x})$ of $\bar{x}(x)$ and its inverse $x(\bar{x})$ exist in S_1 and S_2, respectively. Finally, to deal with the law of transformation of a linear connection and with covariant differentials, we shall assume that the second Fréchet differential $\bar{x}(x; \delta_1 x; \delta_2 x)$ exists in S_1 continuously in x.

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* Presented to the Society, April 11, 1936.

† For the purposes of the present paper alone it is merely necessary to postulate that T is a Fréchet neighborhood space and that the coordinate systems are reciprocal (1-1) transformations.

‡ All the results of the paper continue to hold if we take the *coordinate systems* to be homeomorphisms of Hausdorff neighborhoods onto open subsets $\Sigma \subset S$, where S is a fixed open set in E and is itself a homeomorphic map of some Hausdorff neighborhood.

Since $\bar{x}(x)$ and $x(\bar{x})$ are mutually inverse, we see that $\bar{x}(x; \delta x)$ is a solvable linear function* of δx with $x(\bar{x}; \delta\bar{x})$ as inverse. The symmetry of $\bar{x}(x; \delta_1 x; \delta_2 x)$ in $\delta_1 x$ and $\delta_2 x$ follows from a theorem proved by Kerner.† An application of a theorem‡ of Michal and Elconin on solvable linear functions depending on a parameter shows that the second Fréchet differential $x(\bar{x}; \delta_1\bar{x}; \delta_2\bar{x})$ exists and is continuous in $\bar{x}$ throughout S_2. On differentiating the evident identity $\bar{x}(x; x(\bar{x}; \delta_1\bar{x})) = \delta_1\bar{x}$ we obtain the useful identity

$$\bar{x}(x; \delta_1 x; \delta_2 x) + \bar{x}(x; x(\bar{x}; \delta_1\bar{x}; \delta_2\bar{x})) = 0. \tag{1}$$

The validity of this relation follows on making special use of a theorem§ on the *total* differential of a linear function that depends on a parameter. In the sequel we shall refer to this theorem as Theorem M.

3. *Covariant Differential of a Contravariant Vector Field and the Transformation Law of a Linear Connection.* Let P_0 be any chosen point of the Hausdorff space T. A geometric object whose abstract components undergo the transformation $\bar{\xi} = \bar{x}(x(P_0); \xi)$ will be called a *contravariant vector* (associated with P_0). To avoid long circumlocutions we shall say that ξ *is* a contravariant vector. Similar abbreviations will be made for the components of other geometric objects. That a differential δx is a contravariant vector is clear from the evident formulas

$$\delta\bar{x} = \bar{x}(x; \delta x), \qquad \delta x = x(\bar{x}; \delta\bar{x}).$$

A *contravariant vector field* (c.v.f. for brevity) is a geometric object with abstract components. More precisely, to every Hausdorff neighborhood with coordinate system $x(P)$ there exists a function (called component) $\xi(x)$ on S to E such that in the intersection of two Hausdorff neighborhoods $\bar{\xi}(\bar{x}) = \bar{x}(x; \xi(x))$ under a transformation of coordinates $\bar{x} = \bar{x}(x)$.

* A. D. Michal and V. Elconin, *Differential properties of abstract transformation groups with abstract parameters*, American Journal of Mathematics, vol. 59 (1937), pp. 129–143.

† M. Kerner, Annals of Mathematics, (2), vol. 34 (1933), pp. 546–572. See also A. D. Michal and V. Elconin, *Completely integrable differential equations in abstract spaces* (accepted for publication in the Acta Mathematica).

‡ A. D. Michal and V. Elconin, Acta Mathematica, loc. cit.

§ A. D. Michal, *Postulates for a linear connection*, Annali di Matematica, (1936), pp. 197–220.

Let ξ_1 and ξ_2 be two arbitrary contravariant vectors. The object whose components $\Gamma(x, \xi_1, \xi_2)$ and $\overline{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2)$ are bilinear functions of the vectors will be called a *linear connection** if, in the intersection of two Hausdorff neighborhoods, the components have the law of transformation

$$(2) \qquad \overline{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) + \bar{x}(x; x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2)).$$

With the aid of relation (1) we can immediately rewrite (2) in the equivalent form

$$(3) \qquad \overline{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) - \bar{x}(x; \xi_1; \xi_2).$$

THEOREM 1. *Let the bilinear function $\Gamma(x, \xi_1, \xi_2)$ in the contravariant vectors ξ_1, ξ_2 be the component in the $x(P)$ coordinate system of an object with components. Then, a necessary and sufficient condition that*†

$$(4) \qquad \delta\xi(x) + \Gamma(x, \xi(x), \delta x)$$

be the component in the $x(P)$ coordinate system of a contravariant vector field for every Fréchet differentiable contravariant vector field $\xi(x)$ is that $\Gamma(x, \xi_1, \xi_2)$ be the component of a linear connection.

PROOF. To prove the necessity of the condition we have by hypothesis

$$(5) \qquad \delta\bar{\xi}(\bar{x}) + \overline{\Gamma}(\bar{x}, \bar{\xi}(\bar{x}), \delta\bar{x}) = \bar{x}(x; \delta\xi(x) + \Gamma(x, \xi(x), \delta x)).$$

But, since $\xi(x)$ is a differentiable contravariant vector field and the second differential $\bar{x}(x; \delta_1 x; \delta_2 x)$ exists, it follows that $\delta\bar{\xi}(\bar{x})$ exists and is given by

$$(6) \qquad \delta\bar{\xi}(\bar{x}) = \bar{x}(x; \xi(x); \delta x) + \bar{x}(x; \delta\xi(x)).$$

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* Further generalizations are possible. One can consider still another Banach space E_1 and another type of contravariant vector field with component $V(x)$, a function on E to E_1, subject to a law of transformation $\overline{V}(\bar{x}) = M(x, V(x))$, where $M(x, y)$ is a solvable function of y. The covariant differentials of multilinear forms $F(x, \xi_1, \xi_2, \cdots, \xi_n, V_1, V_2, \cdots, V_i)$ can then be developed with the aid of $\Gamma(x, \xi, \delta x)$ and another kind of linear connection $K(x, V, \delta x)$, a function as EE_1E to E_1, whose law of transformation is

$$\overline{K}(\bar{x}, \overline{V}, \delta\bar{x}) = M(x, K(x, V, \delta x)) - M(x, V; \delta x).$$

† We shall often write the Fréchet differential $f(x; \delta_i x)$ as $\delta_i f(x)$.

We thus obtain immediately the law of transformation

$$\bar{\Gamma}(\bar{x}, \bar{\xi}(\bar{x}), \delta\bar{x}) = \bar{x}(x; \Gamma(x, \xi(x), \delta x)) - \bar{x}(x; \xi(x); \delta x). \tag{7}$$

The sufficiency of the condition follows by a reversal of the steps in the obvious manner.

DEFINITION. *If $\xi(x)$ is a contravariant vector field and $\Gamma(x, \xi, \delta x)$ is a linear connection, then the contravariant vector field* (4) *will be called the covariant differential of $\xi(x)$ and will be denoted by $\xi(x|\, \delta x)$.*

4. *Covariant Differential of Multilinear Forms in Contravariant Vector Fields.* Our treatment of successive covariant differentials will be made to depend on the results of the following fundamental theorem.

THEOREM 2. *Let $\Gamma(x, \xi, \delta x)$ be a linear connection and $F(x, \xi_1, \xi_2, \cdots, \xi_n)$ a function with the following properties:*

(i) *F is a* c.v.f. *valued multilinear form in the n arbitrary contravariant vectors $\xi_1, \cdots, \xi_n$;*

(ii) *the partial Fréchet differential $F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x)$ exists and is continuous in x.*

Then the function $F(x, \xi_1, \xi_2, \cdots, \xi_n|\, \delta x)$ defined by the equation

$$\begin{aligned} F(x, \xi_1, \xi_2, \cdots, \xi_n|\, \delta x) &= F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x) \\ &\quad - \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n) \\ &\quad + \Gamma(x, F(x, \xi_1, \xi_2, \cdots, \xi_n), \delta x) \end{aligned} \tag{8}$$

is a c.v.f. *valued multilinear form in $\xi_1, \xi_2, \cdots, \xi_n, \delta x$. We shall call $F(x, \xi_1, \cdots, \xi_n|\, \delta x)$ the covariant differential of $F(x, \xi_1, \cdots, \xi_n)$.*

PROOF. We shall give here the details of proof only for the case $n = 1$ as the method of proof for the general case differs in no essential manner from that of this special case. By hypothesis, then, we have

$$\bar{F}(\bar{x}, \bar{\xi}) = \bar{x}(x; F(x, \xi)), \tag{9}$$

from which we obtain

$$\bar{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) = \phi(x, \xi; \delta x) - \bar{F}(\bar{x}, \delta\bar{\xi}), \tag{10}$$

where $\phi(x, \xi) = \bar{x}(x; F(x, \xi))$. On using $\bar{\xi} = \bar{x}(x; \xi)$, we find

$$\overline{F}(\bar{x}, \delta\bar{\xi}) = \overline{F}(\bar{x}, \bar{x}(x; \xi; \delta x)). \tag{11}$$

Hence (10) immediately reduces to

$$\overline{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) = \phi(x, \xi; \delta x) - \overline{F}(\bar{x}, \bar{x}(x; \xi; \delta x)). \tag{12}$$

From the laws of transformation (3) and (9), we evidently have

$$\overline{F}(\bar{x}, \overline{\Gamma}(\bar{x}, \bar{\xi}, \delta\bar{x})) = \bar{x}(x; F(x, \Gamma(x, \xi, \delta x))) - \overline{F}(\bar{x}, \bar{x}(x; \xi; \delta x)). \tag{13}$$

From (3) we obtain also

$$\overline{\Gamma}(\bar{x}, \overline{F}(\bar{x}, \bar{\xi}), \delta\bar{x}) = \bar{x}(x; \Gamma(x, F(x, \xi), \delta x)) - \bar{x}(x; F(x, \xi); \delta x). \tag{14}$$

With the aid of (12), (13), (14), and a special use of Theorem M on Fréchet differentials, we finally obtain the law of transformation

$$\overline{F}(\bar{x}, \bar{\xi} \mid \delta\bar{x}) = \bar{x}(x; F(x, \xi \mid \delta x)), \tag{15}$$

where

$$\begin{aligned} F(x, \xi \mid \delta x) = {} & F(x, \xi; \delta x) + \Gamma(x, F(x, \xi), \delta x) \\ & - F(x, \Gamma(x, \xi, \delta x)), \end{aligned} \tag{16}$$

and

$$\overline{F}(\bar{x}, \bar{\xi} \mid \delta\bar{x}) = \overline{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) + \overline{\Gamma}(\bar{x}, \overline{F}(\bar{x}, \bar{\xi}), \delta\bar{x}) - \overline{F}(\bar{x}, \overline{\Gamma}(\bar{x}, \bar{\xi}, \delta\bar{x})).$$

By the Gateaux limit method of evaluating a Fréchet differential and by a theorem of Banach on the limit of a convergent sequence of linear functions, it can readily be shown* that $F(x, \xi; \delta x)$ is a bilinear function of ξ and δx. Hence the covariant differential $F(x, \xi \mid \delta x)$ is bilinear in ξ and δx.

Since the covariant differential of a c.v.f. valued multilinear form in n contravariant vectors is itself a c.v.f. valued multilinear form in $n+1$ contravariant vectors, we may apply Theorem 2 repeatedly and obtain higher order covariant differentials $F(x, \xi_1, \cdots, \xi_n \mid \delta_1 x \mid \cdots \mid \delta_r x)$ of the original multilinear form $F(x, \xi_1, \cdots, \xi_n)$.

* A. D. Michal, Annali di Matematica, loc. cit.

Let $\xi(x)$ be a contravariant vector field with a continuous second differential $\xi(x;\ \delta_1 x;\ \delta_2 x)$ in x and let the linear connection Γ possess a continuous partial differential $\Gamma(x,\ \xi_1,\ \xi_2;\ \delta x)$ in x. If we further assume that the coordinate transformations $\bar{x}(x)$ have continuous third differentials $\bar{x}(x;\ \delta_1 x;\ \delta_2 x;\ \delta_3 x)$ in x, then it can be shown that $x(\bar{x};\ \delta_1 \bar{x};\ \delta_2 \bar{x};\ \delta_3 \bar{x})$ exists and is continuous in $\bar{x}$ and that the above restrictions on $\xi(x)$ and on the linear connection are invariant under such transformations of coordinates. If we then make special use of Theorem M, we find that

$$\xi(x \mid \delta_1 x \mid \delta_2 x) - \xi(x \mid \delta_2 x \mid \delta_1 x) = B(x, \xi(x), \delta_1 x, \delta_2 x) - 2\xi(x \mid \Omega(x, \delta_1 x, \delta_2 x)), \tag{17}$$

where $B(x,\ \xi,\ \delta_1 x,\ \delta_2 x)$, called the *curvature form*, is defined by

$$B(x, \xi, \delta_1 x, \delta_2 x) = \Gamma(x, \xi, \delta_1 x; \delta_2 x) - \Gamma(x, \xi, \delta_2 x; \delta_1 x) + \Gamma(x, \Gamma(x, \xi, \delta_1 x), \delta_2 x) - \Gamma(x, \Gamma(x, \xi, \delta_2 x), \delta_1 x), \tag{18}$$

and where $\Omega(x,\ \delta_1 x,\ \delta_2 x)$, called the *torsion form*, is defined by

$$\Omega(x, \delta_1 x, \delta_2 x) = \frac{1}{2}\left\{\Gamma(x, \delta_1 x, \delta_2 x) - \Gamma(x, \delta_2 x, \delta_1 x)\right\}. \tag{19}$$

It follows from (3) and Kerner's theorem on symmetry of continuous second differentials that the torsion form $\Omega(x,\ \delta_1 x,\ \delta_2 x)$ is a c.v.f. valued bilinear form. Hence with the aid of Theorem 1, Theorem 2, and (17), we see that the curvature form $B(x,\ \xi,\ \delta_1 x,\ \delta_2 x)$ is a c.v.f. valued trilinear form. We can thus form the sequence of covariant differentials of the curvature form $B(x,\ \xi_1,\ \xi_2,\ \xi_3)$:

$$B(x, \xi_1, \xi_2, \xi_3 \mid \delta_1 x \mid \cdots \mid \delta_r x), \qquad (r = 1, 2, \cdots, n).$$

In order that this sequence of covariant differentials exist it is sufficient to assume the existence of $\Gamma(x,\ \xi_1,\ \xi_2;\ \delta_1 x;\ \cdots;\ \delta_{n+1} x)$ and its continuity in x. These restrictions will be invariant under coordinate transformations if we assume that the coordinate transformations $\bar{x}(x)$ possess a continuous differential of order $n+3$ in x.

By a slight variation of the method of proof for Theorem 2, the following theorem can be proved.

THEOREM 3. *If in the hypotheses of Theorem* 2 *we replace* (i) *by the following*:
(i′) *F is an absolute scalar multilinear form in the n arbitrary contravariant vectors $\xi_1, \cdots, \xi_n$,*
then the function $F(x, \xi_1, \cdots, \xi_n | \delta x)$ defined by

$$F(x, \xi_1, \cdots, \xi_n | \delta x) = F(x, \xi_1, \cdots, \xi_n; \delta x) - \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n) \tag{20}$$

is an absolute scalar multilinear form in $\xi_1, \cdots, \xi_n, \delta x$. We shall then call $F(x, \xi_1, \cdots, \xi_n | \delta x)$ the covariant differential of $F(x, \xi_1, \cdots, \xi_n)$.

The conclusion of this theorem continues to hold if the numerical valued form F is replaced by a form F with values in a Banach space.

5. *The "Stokean" Covariant Differential of Alternating Forms.* In this section we do not assume the existence of a linear connection. It is possible nevertheless to obtain with Fréchet differentiation an absolute alternating form of one higher order from a given absolute alternating form in contravariant vectors.

THEOREM 4. *Let $\omega(x, \xi_1, \cdots, \xi_n)$ be a function with the following properties*:
(i) *ω is an absolute scalar multilinear form in the contravariant vectors $\xi_1, \cdots, \xi_n$;*
(ii) *ω is alternating in $\xi_1, \cdots, \xi_n$ (that is, completely skew-symmetric)*;
(iii) *the second partial differential $\omega(x, \xi_1, \cdots, \xi_n; \delta_1 x; \delta_2 x)$ exists and is continuous in x.*
Then under the transformations of coordinates of §3:
(C_1) *the function $\omega(x, \xi_1, \cdots, \xi_n : \delta x)$, called the Stokean covariant differential of $\omega(x, \xi_1, \cdots, \xi_n)$, defined by*

$$\omega(x, \xi_1, \cdots, \xi_n : \delta x) = \omega(x, \xi_1, \cdots, \xi_n; \delta x) - \sum_{i=1}^{n} \omega(x, \xi_1, \cdots, \xi_{i-1}, \delta x, \xi_{i+1}, \cdots, \xi_n; \xi_i), \tag{21}$$

is an alternating absolute scalar multilinear form in $\xi_1, \cdots, \xi_n, \delta x$;

(C_2) $$\omega(x, \xi_1, \cdots, \xi_n : \delta_1 x : \delta_2 x) = 0.$$

It is clear from this theorem that the Cartan-Goursat calculus of alternating forms* can be developed in Hausdorff spaces with Banach coordinates.

In conclusion we note that Theorem 4 continues to hold if the numerically valued form ω is replaced by a form ω with values in a Banach space.

CALIFORNIA INSTITUTE OF TECHNOLOGY

ON A THEOREM OF ENGEL†

BY MAX ZORN

1. *Introduction.* The theorem of Engel which we intend to study in this paper deals with Lie algebras where an identity $(a(a(a \cdots (ab)) \cdots)) = 0$ holds for arbitrary elements a and b. Under various assumptions it has been shown that in this case all products with sufficiently many factors vanish.

The latest result in this direction was a proof,‡ found first by van Kampen, which holds for finite Lie algebras over any field of characterstic zero. The method is *rational*, but it involves the theory of associative algebras and the theory of traces. Another proof of equal generality, with less accent on the theory of traces, has recently been sketched by the writer.§

The new proof to be offered in the present paper dispenses with every apparatus of matrices, traces, and associative systems. It does not presuppose any knowledge about Lie systems. The material advantage of our direct method is the fact that no reference field is required, and that the question of characteristics never enters the discussion.

2. *Definitions.* DEFINITION 1. *A system L of elements a, b, · · · is called a Lie ring (with respect to a commutative ring* P *of*

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* E. Goursat, *Leçons sur le Problème de Pfaff*, 1922; E. Cartan, *Leçons sur les Invariants Intégraux*, 1922; E. Kahler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, 1934.

† Substituted for another paper, which was presented to the Society, June 18, 1936. See the last footnote on this page.

‡ See N. Jacobson, *Rational methods in the theory of Lie algebras*, Annals of Mathematics, vol. 36, p. 875.

§ See this Bulletin, Abstract 42-7-266. (Erroneously the theorem in question is there attributed to Lie.)

scalars ρ) *if addition,* $a+b$, *multiplication,* ab, *and scalar multiplication,* $a\rho=\rho a$, *are defined such that* (i) *the addition defines a group,* (ii) *the multiplication is distributive,* (iii) *scalar multiplication is both distributive and associative,* (iv) $aa=0$, *which implies* $ab+ba=0$, (v) $a(bc)+b(ca)+c(ab)=0$ (Jacobi's identity).

Without loss of generality we assume that P contains ± 1.

DEFINITION 2. *A subset* $S \subset L$ *is a subring if it is closed under the operations of Definition* 1.

DEFINITION 3. *The subring generated from the elements* $a, b, \cdots$ *and the subsets* $A, B, \cdots$ *of* L *is denoted by* $\langle a, b, \cdots, A, B, \cdots \rangle$.

DEFINITION 4. *If* A_i *are subsets* $\subset L$, b *an element* ϵL, *then* A_1+A_2 *is the set of all elements* a_1+a_2, *where* $a_i \epsilon A$; A_1A_2 *the set of all products* a_1a_2; A_1b *the set of all* a_1b; Pb *the set of all scalar multiples* ρb.

In this terminology we may announce, for example, the following theorem.

THEOREM 1. *If* S *is a subring and if* $Sb \subset S$, *then* $\langle S, b\rangle = S+$Pb.

Since multiplication is in general not associative, the position of parentheses is relevant. For the particular purposes of this paper the following conventions are useful.

DEFINITION 5.

$$a_1a_2 \cdots a_{n-1}a_n = a_1(a_2 \cdots a_n) = (a_1(a_2(\cdots (a_{n-1}a_n)) \cdots)),$$

$$a^0b = b,\ a^nb = a(a^{n-1}b) = (a(a(\cdots (a(ab))))),$$

and correspondingly $AB = B$, $A^nB = A(A^{n-1}B)$, *and so forth.*

Because of $aa=0$, we may very well abandon the usual definition of $a^2, \cdots$, so that a^n alone does not denote an element of the ring.

Addition and multiplication of subsets satisfy the distributive law

$$(S+T)(M+N) \subset SM+SN+TM+TN.$$

Iterated application of this distributivity inclusion yields the following result.

THEOREM 2. *The set $(S+T)^nL$ is contained in a sum of products of type*

$$T^{f_0}S^{e_1}\cdots S^{e_k}T^{f_k}L, \tag{1}$$

where $e_i>0$ for $i\leqq k$, $f_i\geqq 0$, $\sum e_i+\sum f_i=n$.

Jacobi's identity, which takes now the form $abc+bca+cab=0$, furnishes the following lemma.

LEMMA 1. *If $ST\subset S$, and $M=-1\cdot M$, then $TSM\subset STM+SM$.*

This shows that a product like $TSTT\cdots TS\cdots SSTL$ is contained in the sum of two others, such that both still contain the same number of factors S, the first has ST instead of TS, and the second has one factor T less. From this we obtain the following lemma.

LEMMA 2. *If $ST\subset S$, then $T^{f_0}S^{e_1}\cdots S^{e_k}T^{f_k}L\subset S^{\Sigma e_i}L$.*

DEFINITION 6. *An element a (a subset A) is nilpotent if for a suitable exponent $a^nL=0$ (respectively, $A^nL=0$).*

CONDITION (1, iii) implies that if an element t is nilpotent, then the set $T=\mathfrak{P}t$ is nilpotent.

3. *Proof of Engel's Theorem.*

LEMMA 3. *If a proper subring S is nilpotent, $S^sL=0$, then there exists an element t which is not contained in S and satisfies $St\subset S$.*

Indeed, since $S^sL=0$ there must be a smallest exponent m such that $S^{m+1}L=S(S^mL)\subset S$. The element t may consequently be taken as an arbitrary element of S^mL which is not contained in S. Such elements do exist, because S is a proper subring and $m\geqq 0$.

If we denote $\mathfrak{P}t$ by T, we have $ST\subset S$ and this (see Theorem 1) makes $S+T$ a ring which contains S as a proper subring.

LEMMA 4. *If S and T are nilpotent, and if $ST\subset S$, then $S+T$ is nilpotent.*

We shall see that if $S^sL=T^tL=0$, this nilpotency is exhibited by $(S+T)^{st}L=0$. On account of Theorem 2, we have only to prove that

$$(2) \quad T^{j_0}S^{e_1}\cdots S^{e_k}T^{j_k}L = 0, \qquad \text{if} \qquad \sum_1^k e_i + \sum_0^k f_i \geqq st.$$

Since the condition $ST \subset S$ for (1) is fulfilled, (2) is certainly true if $\sum e_i \geqq s$. It is also trivially true if one f_i should be $\geqq t$. In any other case we should have at most s exponents f_i; their sum would be at most $(t-1)s$, and the total sum $\sum e_i + \sum f_i$ would be smaller than st. Hence there are no such cases, and Lemma 4 is proved. We are now ready to prove Engel's Theorem in the following general form.

ENGEL'S THEOREM. *If the elements of a subring R are nilpotent, and if in each system of subrings $< R$ there is at least one which is maximal in the system, then R is nilpotent.*

For let S be a maximal nilpotent proper subring of R. Lemma 3 shows that it is contained (properly) in a ring $S+T \subset R$, where $ST \subset S$; $T = Pt$ is nilpotent. Lemma 4 makes $S+T$ nilpotent, and because it is really greater than S it must be the improper subring R. Hence R is nilpotent.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

ON AN INTEGRAL TEST OF R. W. BRINK FOR THE CONVERGENCE OF SERIES

BY C. T. RAJAGOPAL

1. *Introduction.* The test in question is embodied in the following theorem due to R. W. Brink.*

Let $\sum^{\infty} u_n$ be a series of positive terms. Also let $r(x)$ be a function such that (i) $r(n) = r_n = u_{n+1}/u_n$, (ii) $0 < \lambda \leqq r(x) \leqq \mu$, (iii) $r'(x)$ *exists and is continuous,* $\int^{\infty} |r'(x)|\,dx$ *is convergent. Then the convergence of the integral*

$$\int^{\infty} e^{\int^x \log r(t)\,dt}\,dx$$

is necessary and sufficient for the convergence of the series $\sum^{\infty} u_n$.

It is the object of this note to show that Brink's theorem can be expressed in a more general form (Theorem 3 below) which leads at once to all the ratio tests for the convergence of series associated with Kummer's test. The ratio tests are thus welded into unity from a point of view somewhat different from that adopted by Pringsheim in his classical paper *Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern.*†

2. *Connection of Brink's Theorem with the Maclaurin-Cauchy Integral Test.* The problem which confronts us in Brink's theorem is clearly that of setting up an integral $\int^x F(t)dt$ whose behaviour at infinity is reflected by a given series $\sum^{\infty} u_n$. When $\sum^{\infty} u_n$ has all but a finite number of terms positive, the method employed to establish the Maclaurin-Cauchy integral test shows that the convergence of $\int^{\infty} F(x)dx$ is sufficient for that of $\sum^{\infty} u_n$ if for $n \leqq x \leqq n+1$, $0 < u_n \leqq F(x)$, $(n = m, m+1, \cdots)$. Denoting u_{n+1}/u_n by r_n, the condition assumed is that

$$r_{n-1} \cdot r_{n-2} \cdots r_m \leqq \frac{F(x)}{u_m}, \qquad (n \leqq x \leqq n+1),$$

* R. W. Brink, *A new integral test for the convergence and divergence of infinite series*, Transactions of this Society, vol. 19 (1918), p. 188.

† Mathematische Annalen, vol. 35 (1890), pp. 359–372.

that is,

$$\sum_{\nu=m}^{n-1} \log r_\nu \leqq \log \frac{F(x)}{F(m)} + \log \frac{F(m)}{u_m}$$

$$= \int_m^x \frac{F'(t)}{F(t)}\,dt + \log \frac{F(m)}{u_m},$$

or,

$$\sum_{\nu=m}^{n-1}\left[\log r_\nu - \int_\nu^{\nu+1} \frac{F'(t)}{F(t)}\,dt\right] - \int_n^x \frac{F'(t)}{F(t)}\,dt \leqq \log \frac{F(m)}{u_m},$$

$$(n \leqq x \leqq n+1).$$

The right-hand member of the above inequality may be altered to any arbitrary constant; for this would merely imply the multiplication of $F(x)$ by a positive constant in our initial hypothesis. Also, for the truth of the altered inequality the following conditions are sufficient:

(i) $\dfrac{F'(x)}{F(x)}$ is bounded and integrable for $x \geqq m$,

(ii) $\log r_\nu - \displaystyle\int_\nu^{\nu+1} \frac{F'(x)}{F(x)}\,dx \leqq \delta_\nu$,

where $\sum^n \delta_\nu$ is bounded above as $n \to \infty$, which is a consequence of

$$\log r_\nu - \frac{F'(x)}{F(x)} \leqq \delta_\nu, \qquad (\nu \leqq x \leqq \nu+1).$$

If we put $F'(x)/F(x) = f(x)$, the integral whose convergence is sufficient for that of $\sum^\infty u_n$ assumes the form $\int^\infty e^{\int^x f(t)dt}dx$. Further, the divergence of this integral is sufficient for the divergence of $\sum^\infty u_n$ provided that in (ii) above the inequality sign is reversed and $\sum^n \delta_\nu$ is bounded below. Hence we are led to formulate the test as follows.

THEOREM 1. *Let $\sum^\infty u_n$ be a series of positive terms and $r_n = u_{n+1}/u_n$. If*

(i) *$f(x)$ is bounded and integrable for $x \geqq m$, and*

$(\mathcal{C})$: $\int^{\infty} e^{\int^{x} f(t)dt}dx$ *is convergent,*

$\{$*or* $(\mathcal{D})$: $\int^{\infty} e^{\int^{x} f(t)dt}dx$ *is divergent*$\}$;

(ii) *for* $n \leqq x \leqq n+1$,

$(\mathcal{C})$: $\log r_n \leqq f(x) + \delta_n$, $\sum^{n} \delta_n$ *being bounded above,*

$\{$*or* $(\mathcal{D})$: $\log r_n \geqq f(x) + \delta_n'$, $\sum^{n} \delta_n'$ *being bounded below*$\}$;

then $\sum^{\infty} u_n$ *is convergent* $\{$*or divergent*$\}$.

The direct proof of the theorem is exactly on the lines of that of Theorem 2 given below.

BRINK'S INTEGRAL TEST. This is an immediate deduction from Theorem 1. For if $r(x)$ is defined as in Brink's theorem, then

$$\log r(x) - \log r_n = \int_n^x \frac{r'(t)}{r(t)}\,dt,$$

and

$$|\log r(x) - \log r_n| \leqq \frac{1}{\lambda}\int_n^{n+1} |r'(t)|\,dt, \qquad (n \leqq x \leqq n+1).$$

Hence replacing $f(x)$ by $\log r(x)$ and taking

$$\delta_n = \frac{1}{\lambda}\int_n^{n+1} |r'(t)|\,dt, \qquad \delta_n' = -\frac{1}{\lambda}\int_n^{n+1} |r'(t)|\,dt,$$

we see that $\sum^{\infty} u_n$ converges or diverges with

$$\int^{\infty} e^{\int^{x} \log r(t)dt}dx.$$

Thus Theorem 1 includes Brink's integral test, as one of his own theorems in the Annals of Mathematics* includes Hardy's generalization of the Maclaurin-Cauchy integral test†

* R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series*, Annals of Mathematics, (2), vol. 21 (1919–20), p. 41.

† G. H. Hardy, *Theorems connected with Maclaurin's test for the convergence of series*, Proceedings of the London Mathematical Society, (2), vol. 9 (1911).

3. *Preliminary Theorems and Deductions.* Theorem 1 admits of the following generalization.

THEOREM 2. *Let* $\sum^{\infty} u_n$ *be a series of positive terms and* $r_n = u_{n+1}/u_n$. *If*

(i) (D_n) *is a strictly increasing sequence tending to infinity;*

(ii) $d_n \equiv D_n - D_{n-1} = O(1)$;

(iii) $f(x)$ *is bounded and integrable for* $x \geqq D_m$, *and*

$$(\mathcal{C}): \quad \int^{\infty} e^{\int^x f(t)\,dt}\,dx \text{ is convergent},$$

$$\{or \quad (\mathcal{D}): \quad \int^{\infty} e^{\int^x f(t)\,dt}\,dx \text{ is divergent}\};$$

(iv) *for* $D_{n-1} \leqq x \leqq D_n$,

$$(\mathcal{C}): \quad \frac{1}{d_n}\log r_n \leqq f(x) + \delta_n, \ \sum^{n} \delta_\nu d_\nu \text{ being bounded above},$$

$$\left\{or \quad (\mathcal{D}): \quad \frac{1}{d_n}\log r_n \geqq f(x) + \delta_n', \ \sum^{n} \delta_\nu' d_\nu \text{ being bounded below}\right\};$$

then $\sum^{\infty} u_n d_n$ *is convergent* $\{or\ divergent\}$.

PROOF OF CASE $(\mathcal{C})$. Since

$$\frac{1}{d_\nu}\log r_\nu \leqq f(t) + \delta_\nu, \qquad (D_{\nu-1} \leqq t \leqq D_\nu),$$

by integration,

$$\log r_\nu \leqq \int_{D_{\nu-1}}^{D_\nu} f(t)\,dt + \delta_\nu d_\nu.$$

Sum for $\nu = m+1, m+2, \cdots, n-1$; then

$$\log \frac{u_n}{u_{m+1}} \leqq \int_{D_m}^{D_{n-1}} f(t)\,dt + \sum_{\nu=m+1}^{n-1} \delta_\nu d_\nu$$

$$< \int_{D_m}^{D_{n-1}} f(t)\,dt + K_1, \qquad (K_1 \text{ fixed}).$$

Also, for $D_{n-1} \leqq x \leqq D_n$, since $|f(t)| < M$ (fixed), $d_n < K$ (fixed), we have

$$-KM < \int_{D_{n-1}}^{x} f(t)dt.$$

Add the last two inequalities; then

$$\log \frac{u_n}{u_{m+1}} < \int_{D_m}^{x} f(t)dt + K_1 + KM, \quad (D_{n-1} \leqq x \leqq D_n),$$

and

$$u_n < u_{m+1} e^{K_1+KM} e^{\int_{D_m}^{x} f(t)dt}, \qquad (D_{n-1} \leqq x \leqq D_n).$$

Hence, integrating from D_{n-1} to D_n, we have

$$u_n d_n < u_{m+1} e^{K_1+KM} \int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt} dx.$$

Compare the series $\sum^{\infty} u_n d_n$ with the series of positive terms $\sum^{\infty} \int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt} dx$, and the test for convergence $(\mathcal{C})$ follows at once. The test for divergence $(\mathcal{D})$ is similarly proved.

The following is an adjunct to Theorem 2.

THEOREM 2a. *In Theorem 2, suppose the condition $d_n = O(1)$ is dropped and $f(x) < 0$ (that is, the integrand in the test integral is a strictly decreasing function). If other conditions remain the same $\sum^{\infty} u_{n+1} d_n$ is convergent in case $(\mathcal{C})$ and $\sum^{\infty} u_n d_n$ is divergent in case $(\mathcal{D})$.*

A slight modification is required in our former proof of $(\mathcal{C})$:

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{D_n} f(t)dt + K_1.$$

Also, for $D_{n-1} \leqq x \leqq D_n$,

$$0 \leqq -\int_{x}^{D_n} f(t)dt.$$

We add the last two inequalities, and obtain

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{x} f(t)dt + K_1, \qquad (D_{n-1} \leqq x \leqq D_n);$$

whence, as before,

$$u_{n+1}d_n < u_{m+1}e^{K_1}\int_{D_{n-1}}^{D_n} e^{\int_{D_m}^{x} f(t)dt}dx,$$

and the desired result follows by comparison.

DEDUCTIONS FROM THEOREM 2a. (i) Taking $f(x) = -\rho < 0$, $\delta_n = 0$, and putting $u_{n+1}d_n = a_n$, we see that the condition

$$\frac{1}{d_n}\log\frac{a_n \cdot d_{n-1}}{a_{n-1}\cdot d_n} \leqq -\rho < 0$$

is sufficient* for the convergence of $\sum^{\infty} a_n$.

(ii) The condition

$$\frac{1}{d_n}\log\frac{a_n\cdot d_{n-1}}{a_{n-1}\cdot d_n} \leqq -\frac{1}{D_{n-1}} - \frac{1}{D_{n-1}\cdot l_1 D_{n-1}} - \cdots - \frac{\alpha}{D_{n-1}\cdot l_1 D_{n-1}\cdots l_p D_{n-1}}, \qquad (\alpha > 1),$$

where $l_1 D_{n-1} = \log D_{n-1}$, $l_2 D_{n-1} = \log\log D_{n-1}$, $\cdots$ (and $n \geqq m+1$ which is such that $l_p D_m > 0$), is sufficient for the convergence of $\sum^{\infty} a_n$. For this implies that

$$\frac{1}{d_n}\log\frac{u_{n+1}}{u_n} \leqq -\frac{1}{x} - \frac{1}{x\cdot l_1 x} - \cdots - \frac{\alpha}{x\cdot l_1 x\cdots l_p x}, \qquad (\alpha > 1;\ D_{n-1} \leqq x \leqq D_n).$$

Hence taking

$$f(x) = -\frac{1}{x} - \frac{1}{x\cdot l_1 x} - \cdots - \frac{\alpha}{x\cdot l_1 x\cdots l_p x},\ (\alpha > 1);\ \delta_n = 0,$$

we deduce the convergence of $\sum^{\infty} a_n$.

(iii) Similarly the condition

$$\frac{1}{d_n}\log\frac{a_{n+1}\cdot d_n}{a_n\cdot d_{n+1}} \geqq -\frac{1}{D_n} - \frac{1}{D_n\cdot l_1 D_n} - \cdots - \frac{\alpha}{D_n\cdot l_1 D_n\cdots l_p D_n}, \qquad (\alpha \leqq 1),$$

is sufficient for the divergence of $\sum^{\infty} a_n$.

Setting $D_n = n$ in (ii) and (iii), we obtain Bertrand's logarithmic criteria for convergence and divergence.

* A. Pringsheim, loc. cit., p. 370.

4. *Generalization of Brink's Theorem.*

THEOREM 3. *Let $\sum^{\infty} a_n$ be a series of positive terms. If*

(i) *(D_n) is a strictly increasing sequence tending to infinity;*

(ii) *$d_n \equiv D_n - D_{n-1} = O(1)$;*

(iii) *$f(x)$ has a continuous derivative $f'(x)$ and $\int^{\infty} |f'(x)|\,dx$ is convergent;*

(iv) $(\mathcal{C})$: $\int^{\infty} e^{\int^x f(t)\,dt}\,dx$ *is convergent,*

$\{$*or,* $(\mathcal{D})$: $\int^{\infty} e^{\int^x f(t)\,dt}\,dx$ *is divergent*$\}$;

(v) $(\mathcal{C})$: $\dfrac{1}{d_n} \log \dfrac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \leqq f(D_n)$,

$\{$*or,* $(\mathcal{D})$: $\dfrac{1}{d_n} \log \dfrac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \geqq f(D_n)\}$;

then $\sum^{\infty} a_n$ is convergent $\{$*or divergent*$\}$.

PROOF OF $(\mathcal{C})$. Denoting a_n/d_n by u_n, we have in the notation of Theorem 2,

$$\frac{1}{d_n} \log r_n \leqq \int_x^{D_n} f'(t)\,dt + f(x)$$

$$\leqq \int_{D_{n-1}}^{D_n} |f'(t)|\,dt + f(x), \qquad (D_{n-1} \leqq x \leqq D_n).$$

Whence, choosing $\delta_n = \int_{D_{n-1}}^{D_n} |f'(t)|\,dt$ in Theorem 2, we deduce the convergence of $\sum^{\infty} u_n d_n \equiv \sum^{\infty} a_n$.

Proof of $(\mathcal{D})$ is similar.

DEDUCTIONS FROM THEOREM 3. (i) If

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} = f(D_n),$$

then, under the conditions assumed, the convergence of

$$\int^{\infty} e^{\int^x f(t)\,dt}\,dx$$

is necessary and sufficient for the convergence of $\sum^{\infty} a_n$. When $D_n = n$, we have Brink's theorem.

(ii) Taking $f(x) = -\rho < 0$, we see that the condition

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \leqq -\rho < 0$$

is sufficient for the convergence of $\sum^{\infty} a_n$.* Since $\log\gamma \leqq \gamma - 1$, $(\gamma > 0)$, it follows that the above condition can also be expressed in Kummer's form:†

$$\frac{1}{d_n}\left(\frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1\right) \leqq -\rho < 0.$$

(iii) Taking

$$f(x) = -\frac{1}{x} - \frac{1}{x \cdot l_1 x} - \cdots - \frac{\alpha}{x \cdot l_1 x \cdots l_p x}, \quad (\alpha > 1),$$

we observe that the condition

$$\left.\begin{aligned} &\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \\ \text{or,}\quad &\frac{1}{d_n}\left(\frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1\right) \end{aligned}\right\} \leqq -\frac{1}{D_n} - \frac{1}{D_n \cdot l_1 D_n} - \cdots - \frac{\alpha}{D_n \cdot l_1 D_n \cdots l_p D_n}, \quad (\alpha > 1),$$

is sufficient for the convergence of $\sum^{\infty} a_n$.

The corresponding divergence criterion has already been given.

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* A. Pringsheim, loc. cit., p. 371.

† A. Pringsheim, loc. cit., p. 361, footnote.

ON A CLASS OF RECURRENT SEQUENCES*

BY H. E. ROBBINS

The first example of a recurrent, non-periodic sequence was given by Marston Morse† in connection with the geodesics on a surface of negative curvature. A discussion of their significance, together with a general method of definition may be found in Birkhoff's *Dynamical Systems*, 1927, p. 246. In this note they will be considered independently of their origin, and a class of such sequences with certain interesting properties will be defined. (Theorem 1 of the present paper, except insofar as it refers to the particular sequence under discussion, is therefore not new.) As a preliminary to this we shall make certain definitions.

A *sequence* is a doubly-infinite row of the symbols 1 and 2: $\cdots c_{-3}c_{-2}c_{-1}c_0c_1c_2c_3\cdots$, where each c_i is either a 1 or a 2. A *block* is a set of consecutive members of a sequence: $c_mc_{m+1}\cdots c_n$ and has length s if there are s symbols in it. A sequence is *periodic* if there exists an integer p such that $c_{n+p}=c_n$ for all n. A sequence is *recurrent* if there exists a function of integers $f(n)$ with integral values, such that any block of length n chosen anywhere in the sequence is contained as a block in any block of length $f(n)$. The least such function $f(n)$ will be called the *ergodic function* of the sequence.

As an example of such a sequence, let

$$a_0 = 12, \quad a_1 = a_0^{-1}a_0a_0 = 21\underline{12}12, \quad \cdots, \quad a_{n+1} = a_n^{-1}a_na_n, \quad \cdots.$$

To define our sequence we number the symbols starting with the original

$$\cdots c_0c_1\cdots$$
$$\cdots 12\cdots$$

which has been underlined above. We may omit a more precise definition of the nth symbol since it will be unnecessary for our present purpose. We note that

* Presented to the Society, September 1, 1936.

† Transactions of this Society, vol. 22 (1921), p. 94.

$$(a_n^{-1}a_n a_n)^{-1} = a_n^{-1}a_n^{-1}a_n,$$

where, as above, a_n^{-1} denotes the symbols of a_n in reverse order.

THEOREM 1. *The sequence thus defined is recurrent but not periodic.*

PROOF OF RECURRENCE. First of all we note that there are $2 \cdot 3^n$ symbols in the nth stage of the construction, a_n. Now let us choose any block of length s:

$$c_{m+1}c_{m+2} \cdots c_{m+s}, \tag{1}$$

and let n be the smallest integer such that $2 \cdot 3^n \geqq s$.

Since for any m the sequence may be regarded as built up of a succession of blocks a_m and a_m^{-1} in some order, it follows that the block (1) must be contained in one of the following blocks:

$$a_n a_n, \quad a_n a_n^{-1}, \quad a_n^{-1}a_n, \quad a_n^{-1}a_n^{-1}. \tag{2}$$

But

$$a_{n+1} = a_n^{-1}a_n a_n, \qquad a_{n+1}^{-1} = a_n^{-1}a_n^{-1}a_n,$$

$$a_{n+2} = a_{n+1}^{-1}a_{n+1}a_{n+1} = a_n^{-1}a_n^{-1}a_n a_n^{-1}a_n a_n a_n^{-1}a_n a_n,$$

$$a_{n+2}^{-1} = a_n^{-1}a_n^{-1}a_n a_n a_n^{-1}a_n^{-1}a_n a_n a_n^{-1}a_n a_n,$$

and *each* of the blocks (2) is contained in *both* a_{n+2} and a_{n+2}^{-1}. It follows that the block (1) is contained in *any* block of length twice the length of a_{n+2}, that is, of length $4 \cdot 3^{n+2}$. So if we set $f(s) = 4 \cdot 3^{n+2}$, the sequence is recurrent with ergodic function at most equal to $f(s)$. To express $f(s)$ explicitly in terms of s, or rather, to get an upper bound for it, we proceed as follows. By definition, n is the smallest integer such that $2 \cdot 3^n \geqq s$, that is, such that $n \geqq \log_3(s/2)$. Therefore, $n < \log_3(s/2) + 1$. It follows that

$$f(s) \leqq 4 \cdot 3^{n+2} < 4 \cdot 3^{\log_3(s/2)+3} = 4 \cdot \frac{s}{2} \cdot 27 = 54 \cdot s,$$

that is, $f(s) < 54 \cdot s$.*

PROOF OF NON-PERIODICITY. It is easily seen that periodic sequences have ergodic functions of the form $f(n) = n + c$, where

* By a more detailed analysis of the sequence, a much lower bound can be found for the ergodic function $f(s)$.

c is some constant. We shall show that the ergodic function of our sequence is not of this form. The proof will be based on the following lemma.

LEMMA 1. *For all $n=1, 2, 3, \cdots$ the block a_n^{-1} is not contained in $a_n a_n$ or in $a_n a_n^{-1}$ or $a_n^{-1}a_n$, except as the initial or final block in the latter two.*

PROOF.

$$a_1 a_1 = 211212211212,$$
$$a_1^{-1}a_1 = 212112211212, \qquad a_1^{-1} = 212112,$$
$$a_1 a_1^{-1} = 211212212112,$$

and the assertion is obvious on inspection. Now suppose the assertion true for $n=m$. By definition

$$a_{m+1} = a_m^{-1}a_m a_m, \qquad a_{m+1}^{-1} = a_m^{-1}a_m^{-1}a_m,$$
$$a_{m+1}a_{m+1} = a_m^{-1}a_m a_m a_m^{-1}a_m a_m,$$
$$a_{m+1}^{-1}a_{m+1} = a_m^{-1}a_m^{-1}a_m a_m^{-1}a_m a_m,$$
$$a_{m+1}a_{m+1}^{-1} = a_m^{-1}a_m a_m a_m^{-1}a_m^{-1}a_m,$$

and inspection again shows the truth of the assertion for $n=m+1$. Now to show non-periodicity, we choose some $n=1, 2, 3, \cdots$ such that $2\cdot 3^n > c$, where c is an arbitrary positive integer. The blocks a_n^{-1} and $a_n a_n$ both occur in the sequence, but the length of $a_n^{-1}=2\cdot 3^n$, the length of $a_n a_n = 4\cdot 3^n > 2\cdot 3^n + c$, while $a_n a_n$ contains no sub-block of the form a_n^{-1} by the lemma. Thus our sequence cannot be periodic. This concludes the proof of Theorem 1. We may state explicitly the following corollary.

COROLLARY. *There exist recurrent, non-periodic sequences whose ergodic functions are bounded above by linear functions (without constant terms).*

Having obtained a recurrent, non-periodic sequence whose ergodic function is bounded above by a linear function and which is thus, in a sense, close to the periodic case, we turn our attention to the other extreme and ask whether there exist r. n. p. sequences whose ergodic functions increase arbitrarily rapidly. We may state the problem as follows:

Given a function $R(n)$ (of integers, with integral values) such that

$$R(n) > n, \tag{3}$$

does there exist an r. n. p. sequence with ergodic function $f(n)$ such that for each n greater than some integer d, $f(n) > R(n)$; that is, does there exist a block of length $\leqq n$ and a block of length $> R(n)$ which does not contain it as a sub-block?

We shall see that this question is to be answered in the affirmative. Let us define a sequence as follows:

$$a_0 = 12, \qquad a_1 = a_0^{-1}\underbrace{a_0 a_0 \cdots a_0}_{R(6)\text{ times}}, \cdots, a_{n+1} = a_n^{-1}\underbrace{a_n a_n \cdots a_n}_{R[3\cdot l(a_n)]\text{ times}},$$

where $l(a_n)$ denotes the number of symbols in a_n. The precise method we choose to number the symbols of the sequence is of course arbitrary, but we may start with the "original" $\cdots 12 \cdots$ as in the previous case. That this sequence is r. n. p. follows by much the same reasoning as in the previous simpler case. We shall confine our attention to proving that the ergodic function increases rapidly enough. The proof will be based on the following lemma.

LEMMA 2. *For any $n \geqq 1$, the block $a_n a_n \cdots a_n$ of arbitrary length contains no sub-block of the form a_n^{-1}.*

PROOF. It will be sufficient to show that this is true for the block $a_n a_n$. The assertion holds for $n = 1$:

$$a_1 = a_0^{-1}\underbrace{a_0 a_0 \cdots a_0}_{} = 21\ \underbrace{12\ 12 \cdots 12}_{},$$

$$a_1 a_1 = 21\ \underbrace{12\ 12 \cdots 12}_{}\ 21\ \underbrace{12\ 12 \cdots 12}_{},$$

$$a_1^{-1} = \underbrace{21\ 21 \cdots 21}_{}\ 12.$$

Now assume it to hold for $n = m$. By definition,

$$a_{m+1}a_{m+1} = a_m\underbrace{a_m \cdots a_m}_{R[3\cdot l(a_m)]}a_m^{-1}\underbrace{a_m \cdots a_m}_{R[3\cdot l(a_m)]},$$

$$a_{m+1}^{-1} = \underbrace{a_m^{-1}a_m^{-1} \cdots a_m^{-1}}_{R[3\cdot l(a_m)]}a_m,$$

and from (2) the assertion is seen to hold for $n=m+1$, proving the lemma.

Now to prove the theorem, we choose an arbitrary integer s, where $l(a_n) \leqq s \leqq l(a_{n+1})$, $(n=1, 2, 3, \cdots)$, and suppose that $p \cdot l(a_n) \leqq s \leqq (p+1) \cdot l(a_n)$, where $1 \leqq p \leqq R[3 \cdot l(a_n)]$. We observe that

$$a_{n+1} = a_n^{-1} \underbrace{a_n \cdots a_n}_{R[3 \cdot l(a_n)] \text{ times}},$$

$$a_{n+2} = a_{n+1}^{-1} \underbrace{a_{n+1} \cdots a_{n+1}}_{R[3 \cdot l(a_{n+1})] \text{ times}},$$

$$a_{n+2}^{-1} = \underbrace{[a_n^{-1} \cdots a_n^{-1} a_n][a_n^{-1} \cdots a_n^{-1} a_n] \cdots [a_n^{-1} \cdots a_n^{-1} a_n] \cdots}_{R[3 \cdot l(a_{n+1})] \text{ blocks}}.$$

The number of symbols in this initial portion of a_{n+2}^{-1} is equal to $R[3 \cdot l(a_{n+1})] \cdot [R(3 \cdot l(a_n)) + 1] \cdot l(a_n)$, while it contains no sub-block of the form $a_n a_n a_n$ by Lemma 2. Thus, if $p \geqq 3$, the theorem is proved. If $p < 3$, that is, $l(a_n) \leqq s \leqq 3 \cdot l(a_n)$, we simply observe that the block $a_n a_n \cdots a_n$ which is contained in a_{n+1} contains no sub-block of the form a_n^{-1} by the lemma. But its length is $> R[3 \cdot l(a_n)]$, which completes the proof of Theorem 2.

THEOREM 2. *There exist recurrent, non-periodic sequences whose ergodic functions increase arbitrarily rapidly.*

HARVARD UNIVERSITY

NOTE ON A CERTAIN RING-CONGRUENCE

BY H. S. VANDIVER

1. *Introduction.* Consider the functions

$$\alpha_1 a_1^n + \alpha_2 a_2^n + \cdots + \alpha_k a_k^n = f_n(\alpha_1, \cdots, \alpha_k),$$

where the a's are rational integers and the α's belong to a ring R including the rational integers. Further, for any a_i prime to m, let

$$a_i^d \equiv 1 \pmod{m}, \qquad (i = 1, 2, \cdots, k).$$

Now we set up the function

$$\alpha_1 a_1^n x^{a_1^d} + \alpha_2 a_2^n x^{a_2^d} + \cdots + \alpha_k a_k^n x^{a_k^d} = f_n(x) = f_n(x, \alpha_1, \cdots, \alpha_k).$$

Consider the operation of differentiating $f_n(x)$ with respect to x and then multiplying the result by x. We shall call this operation $E(f)$. Similarly we shall call $E^{(j)}(f)$ the result of carrying out this operation j times on f. Hence

$$E^{(j)} f_n(x) = f_{n+jd}(x), \tag{1}$$

and

$$\left[E^{(j)} f_n(x)\right]_{x=1} = f_{n+jd}(\alpha_0, \alpha_1, \cdots, \alpha_k). \tag{2}$$

Now consider any function of the form

$$H(x) = \sum_h \gamma_h x^h,$$

where the γ's are in R and the summation ranges over any finite number of rational integers, h. If u_1 and u_2 are functions of this type, then it may easily be shown by induction that

$$E^{(j)}(u_1 u_2) = (u_1 + u_2)^j,$$

where on the right we expand by the binomial theorem and replace $(u_1)^t$ by $u_1^{(t)}$ with $u_1^{(t)} = E^{(t)}(u_1)$ and similarly for $(u_2)^s$, with $(u_1)^0 = u_1$; $(u_2)^0 = u_2$. In fact, this scheme corresponds to setting $x = e^v$, where e is the Napierian base, and differentiating $u_1 u_2$, j times with respect to v, if we should assume that R contains the field of all real numbers. More generally we have

$$E^{(j)}(u_1u_2\cdots u_s) = (u_1 + u_2 + \cdots + u_s)^j, \tag{3}$$

where, in the expression on the right, we expand by the multinomial theorem and replace u_i^r by $u_i^{(r)}$ with $u_i^{(r)} = E^{(r)}(u_i)$; the latter theorem is written in the form

$$(u_1 + u_2 + \cdots + u_s)^j = \sum \frac{j!}{c_1!c_2!\cdots c_s!} u_1^{c_1}u_2^{c_2}\cdots u_s^{c_s}, \tag{4}$$

the summation ranging independently over each set of positive or zero c's satisfying

$$c_1 + c_2 + \cdots + c_s = j,$$

and further $u_i^0 = u_i$.

2. *The Main Theorem.* Write

$$f_{n_i}^{(i)}(x) = \sum_{r=1}^{k_i} \alpha_{ri}a_{ri}^{n_i}x^{a_{ri}^d}. \tag{5}$$

If $(b_1, b_2, \cdots, b_v)$ is the greatest common divisor of $b_1, b_2, \cdots, b_v$, consider the a_{ri}^d's in (5) which have factors in common with m and let l_i be the greatest common divisor of all such. Consider the product

$$f_{n_1}^{(1)}(x^{\beta_1})f_{n_2}^{(2)}(x^{\beta_2})\cdots f_{n_s}^{(s)}(x^{\beta_s}) = F, \tag{6}$$

where the β's are integers such that

$$\beta_1 + \beta_2 + \cdots + \beta_s \equiv 0 \pmod{m}. \tag{6a}$$

We now proceed to carry out in two different ways the operation $E^{(j)}(F)$ and finally set $x=1$ in each result. Employing (2) and (3), we find

$$[E^{(j)}(F)]_{x=1} = (f_{n_1} + f_{n_2} + \cdots + f_{n_s})^j,$$

where, after expansion of the right-hand member following (3), we set

$$f_{n_i}^t = \beta^t f_{n_i+td}^{(i)}(\alpha_1, \alpha_2, \cdots, \alpha_{k_i}).$$

Consider a term in $f_{n_i}^{(i)}(x^{\beta_i})$ in which a_{gi} is prime to m,

$$\alpha_{gi}a_{gi}^{n_i}x^{a_{gi}^d\beta_i}.$$

Set $a_{\varrho i}^{d}=1+mq(a_{\varrho i})$; then the above becomes

$$\alpha_{\varrho i}a_{\varrho i}^{n_i}x^{\beta_i+\beta_i mq(a_{\varrho i})}.$$

The terms in our f in which the $a_{\varrho i}$'s are prime to m may then be written

$$G_i \equiv x^{\beta_i}\sum_{\varrho}\alpha_{\varrho i}a_{\varrho i}^{n_i}x^{\beta_i mq(a_{\varrho i})},$$

so that

$$f_{n_i}^{(i)}(x^{\beta_i}) = G_i + l_i C(x),$$

where $C(x)$ is a function of the same type as $H(x)$. Since

$$\beta_1+\beta_2+\cdots+\beta_s \equiv 0 \pmod{l},$$

then

$$\prod_{i=1}^{s} G_i$$

can be expressed as the sum of terms of the form $Ax^{m\gamma}$, where A belongs to R. Hence we may write

$$F \equiv \sum A_\gamma x^m + LD(x),$$

where $L=(l_1, l_2, \cdots, l_s)$; $D(x)$ is of the same type as $H(x)$, and then

$$E^{(j)}\left[Ax^{m\gamma}\right]_{x=1} \equiv 0 \pmod{m^j},$$

and also, if we write (mod L, m^j) for (mod (L, m^j)),

$$\left[E^{(j)}(F)\right]_{x=1} \equiv 0 \pmod{L, m^j}.$$

THEOREM. *Let R be a ring containing the ring of rational integers. Put*

$$f_{n_i}^{(i)}(\alpha_1, \alpha_2, \cdots, \alpha_{k_i}) = \sum_{r=1}^{k_i}\alpha_{ri}a_{ri}^{n_i},$$

where the a's are rational integers and the α's belong to R. Further, let

$$a_{ri}^{d} \equiv 1 \pmod{m}, \qquad (i=1, 2, \cdots, k);$$

let l_i be the greatest common divisor of all the $a_{ri}^{n_i}$ in the above which

have factors in common with m; and let $\beta_1, \beta_2, \cdots, \beta_s$ be rational integers such that

$$\beta_1 + \beta_2 + \cdots + \beta_s \equiv 0 \pmod{m}.$$

Then

$$(7) \qquad (f_{n_1} + f_{n_2} + \cdots + f_{n_s})^j \equiv 0 \pmod{m^j, l_1, l_2, \cdots, l_s},$$

where we expand the left-hand member, employing (4), *and set*

$$f^t_{n_i} = \beta^t f^{(i)}_{n_i+td}(\alpha_1, \alpha_2, \cdots, \alpha_{k_i}), \qquad (i = 1, 2, \cdots, s).$$

3. *Applications of the Theorem.* The above general theorem has many applications, some of which will be considered here. Kummer* gave a result which may be expressed as follows:

$$(8) \quad h^n(h^{p-1} - 1)^j \equiv 0 \pmod{p^j}, \ (n - 1 \geqq j; n \not\equiv 0 \pmod{p - 1}),$$

where p is an odd prime; the left-hand member is expanded in full, then b_t/t is substituted for h^t, and the b's are defined by the recursion formula

$$(b + 1)^n = b_n, \qquad (n > 1),$$

in which we expand the left-hand member by the binomial theorem and substitute b_k for b^k. The latter formula gives the Bernoulli numbers.

To apply the main theorem in the present paper to Bernoulli numbers, we employ the known formula

$$S_i(p^k) = 1^i + 2^i + \cdots + (p^k - 1)^i \equiv p^k b_i \pmod{p^{2k}},$$

where i is even and $p > 3$. We also employ the formula

$$\frac{(n^i - 1)S_i(p^\alpha)}{p^\alpha} = \sum_{a=1}^{p^\alpha - 1} \sum_{s=1}^{i} a^i C_{s,i} \left(\frac{y_a}{a}\right)^s p^{\alpha(s-1)},$$

where n is prime to p and

$$y_a \equiv -\frac{a}{p} \pmod{n}, \qquad (0 \leqq y_a < n).$$

These give

$$\frac{n^{2i} - 1}{2i} b_{2i} \equiv \sum_{a=1}^{p^\alpha - 1} y_a a^{2i-1} \pmod{p^\alpha},$$

* Journal für Mathematik, vol. 41 (1851), pp. 368–372.

which we immediately connect up with the f-functions treated in our theorem, and the latter gives

$$h_1^{n_1}h_2^{n_2}\cdots h_s^{n_s}(\beta_1 h_1^{p-1}+\beta_2 h_2^{p-1}+\cdots+\beta_s h_s^{p-1})^j \equiv 0 \pmod{p^j, p^{n_1-1}, p^{n_2-1}, \cdots, p^{n_s-1}},$$
$$(n_i \not\equiv 0 \pmod{p-1};\ i=1, 2, \cdots, s),$$

where the left-hand member is expanded in full and b_t/t substituted for h_i^t in the result, $(i=1, 2, \cdots, s)$. To obtain (8) from (7), set $s=2$, and

$$f_{n_1}=\sum_{a=1}^{p^j-1} y_a a^{n-1}, \qquad f_{n_2}=1, \beta_1=1, \beta_2=-1,$$

and the result follows.

Frobenius* gave the relation

$$H^a(1-H^b)^c \equiv 0 \pmod{(p^a, p^{ec})}, \tag{10}$$

where p is a prime, b is a multiple of $p^{e-1}(p-1)$, and the left-hand member is expanded in full and H^t is replaced by H_t. Further, H_t is defined by the recursion formula

$$(H+1)^n = xH^n, \qquad (n>0),$$

where the left-hand member is expanded by the binomial theorem and H^t is replaced by H_t. This gives H as the quotient of two polynomials in x with rational integral coefficients. If these fractions are expressed in their lowest terms, the numerators are called Euler polynomials. Each denominator is a power of $(x-1)$. The relation (10) can be obtained from (7) if we take R as the polynomial ring obtained by adjoining the indeterminate x to the rational ring and extending the result given by Frobenius† so that we have the congruence mod p^j which is analogous to the one he gives mod p. The $R_n(x)$ referred to in this formula is defined by

* Berliner Mathematische Gesellschaft, Sitzungsberichte, 1910, p. 826 and p. 841.

† Loc. cit., p. 843, relation (1).

$$H^n(x) = \frac{R_n(x)}{(x-1)^n}.$$

The relation (7) gives many generalizations of (9). For example we can take $m = p^s$ in lieu of $m = p$. Further details I hope to give in another paper on Bernoulli numbers and Euler polynomials.

UNIVERSITY OF TEXAS

A THEOREM ON MEAN RULED SURFACES

BY MALCOLM FOSTER

Consider the ruled surface formed by the normals to a surface S along some curve C on S. We ask: What are the curves C for which the line of striction of the ruled surface is the locus of the centers of mean curvature corresponding to C?

On S we take the lines of curvature parametric. Referred to the moving trihedral of S, the direction-cosines of the normal are (0, 0, 1), and the variations in these are given by*

$$dX = qdu, \qquad dY = -p_1 dv, \qquad dZ = 0.$$

Now the displacement of the central point on each generator of the ruled surface is orthogonal both to the normal and to its neighboring position. Hence we have

$$\delta z = 0, \qquad qdu\delta x - p_1 dv\delta y + \delta z = 0,$$

which reduce to

$$(1) \qquad qdu(\xi du + zqdu) - p_1 dv(\eta_1 dv - zp_1 dv) = 0.$$

If in (1) we assign a value to the ratio dv/du, this equation will determine the distance z to the line of striction on the ruled surface defined by this ratio; and if to z we assign a given value, equation (1) will determine the curves, (though not necessarily real), for which this assigned value of z is the distance to the lines of striction.

From (1) we have for the problem at hand,

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* Eisenhart, *Differential Geometry of Curves and Surfaces*, pp. 166–174.

$$z = \frac{p_1\eta_1 dv^2 - q\xi du^2}{p_1^2 dv^2 + q^2 du^2} = \frac{q\eta_1 - p_1\xi}{2p_1 q},$$

which after some simplification may be written

$$(2) \qquad (q\eta_1 + p_1\xi)(p_1^2 dv^2 - q^2 du^2) = 0.$$

It is readily seen that the vanishing of the first factor in (2) would mean that S is either a sphere or a plane. Hence, excluding these cases, the equation of the curves defining the mean ruled surfaces of the congruence of normals to S is given by*

$$(3) \qquad q^2 du^2 - p_1^2 dv^2 = 0.$$

Since this is identical with the equation which defines the principal surfaces of the Ribaucour congruence for which S is the director surface, we have the following theorem.

THEOREM. *The mean ruled surfaces of the congruence of normals to a surface S are represented on S by curves which define also the principal surfaces of the Ribaucour congruence for which S is the director surface.*

WESLEYAN UNIVERSITY

* Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), p. 177.

A NOTE ON THE RELATION BETWEEN INTEGRAL AND TCHEBYCHEFF APPROXIMATION BY POLYNOMIALS IN THE COMPLEX DOMAIN*

BY W. E. SEWELL

1. *Introduction.* Let C be a rectifiable Jordan curve in the z-plane and let R be the limited simply connected region bounded by C. Let $f(z)$ be analytic in R and continuous in $R+C\equiv\overline{C}$, and let $P_n(z)$ be a polynomial of degree n in z which minimizes the integral

$$\int_C w(z)\,|\,f(z)-P_n(z)\,|^p\,|\,dz\,|, \tag{1}$$

where p is a fixed positive number, and $w(z)$ is a bounded non-negative measurable function bounded from zero; the existence of such a polynomial $P_n(z)$ is well known.† Dunham Jackson‡ has given an evaluation for $|f(z)-P_n(z)|$, z in $\overline{C}$, with various restrictions on C and on R. In this note we sharpen these results for curves with corners, and extend them to more general smooth curves and to arbitrary rectifiable Jordan curves.

We also consider the case $p=2$ in particular and the development of $f(z)$ of class L^2 in normal and orthogonal polynomials. Our two principal results are the following theorems.

THEOREM A. *Let C be a rectifiable Jordan curve in the z-plane and let $f(z)$ be analytic in C and continuous in $\overline{C}$. Let $p_n(z)$ be an arbitrary polynomial of degree n such that $|f(z)-p_n(z)|\leqq\epsilon_n$, z in $\overline{C}$. Then $|f(z)-P_n(z)|\leqq Mn^{2/p}\epsilon_n$, $p>0$, z in $\overline{C}$, where M is a constant independent of n and z, and $P_n(z)$ is a polynomial of degree n which minimizes* (1).

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* Presented to the Society, December 31, 1936.

† See, for example, J. L. Walsh, *Interpolation and Approximation*, Colloquium Publications of this Society, vol. 20, 1935, pp. 351–352.

‡ *On certain problems of approximation in the complex domain*, this Bulletin, vol. 36 (1930), pp. 851–857; *On the application of Markoff's theorem to problems of approximation in the complex domain*, this Bulletin, vol. 37 (1931), pp. 883–890. These papers will be referred to hereafter as JI and JII, respectively.

THEOREM B. *Let C be a rectifiable Jordan curve in the z-plane and let $f(z)$ belong to L^2 on C. Let $\{P_n(z)\}$ be the set of polynomials normal and orthogonal on C and let**

$$a_k = \int_C f(z)\overline{P_k(z)}\,|\,dz\,|.$$

If $\sum_{k=0}^{\infty}|a_k|\,k$ converges, the function $f_1(z) \equiv \sum_{k=0}^{\infty} a_k P_k(z)$ is analytic in C, continuous in $\overline{C}$, and $f_1(z) = f(z)$ almost everywhere on C.

The method is an application of recent results of the author† on the modulus of the derivative of a polynomial and is the same as that used by Jackson in JI and JII.

It should be noted here that in §3 (Theorem A) the region may be a multiply connected region bounded by a finite number of rectifiable Jordan curves, or made up of a finite number of separate regions of similar character.‡

2. *Jordan Curves and Derivatives of Polynomials.*§ Let R, with boundary C, be a limited simply connected region in the z-plane and let $z = \psi(w)$ map K, the complement (with respect to the extended plane) of $\overline{C}$, on $|w| > 1$ so that the points at ∞ in the two planes correspond to each other. We will say that C *is a curve of Type S*‖ *if*

$$0 < N_1 < \left| \frac{\psi(w_1) - \psi(w_2)}{w_1 - w_2} \right| < N_2 < \infty, \tag{2}$$

$$(|w_1| \geqq 1,\ |w_2| \geqq 1),$$

where N_1 and N_2 are constants independent of w_1 and w_2. If C is a curve of Type S it is shown in SII *that for $P_n(z)$, an arbitrary polynomial of degree n, the inequality $|P_n(z)| \leqq M$, z on C, im-*

* $\bar{a}$ denotes the conjugate of the complex number a.

† *On the modulus of the derivative of a polynomial*, this Bulletin, vol. 42 (1936), pp. 699–702; *Generalized derivatives and approximation by polynomials*, Transactions of this Society, vol. 41 (1937), pp. 84–123. These papers will be referred to hereafter as SI and SII, respectively.

‡ See JII, p. 885.

§ The results given here are extensions of Bernstein's and Markoff's Theorems on the moduli of the derivatives of polynomials; see SII for references.

‖ For the geometric properties of C see W. Seidel, *Über die Ränderzuordnung bei konformen Abbildungen*, Mathematische Annalen, vol. 104 (1931), pp. 182–243; especially pp. 217–221.

*plies** $|P_n'(z)| \leqq MM_1 n$, *z on C; here, and below M and* M_1 *are constants independent of n and z; the constant* M_1 *depends on C.*

For curves with corners we need the following definition (see SII).

DEFINITION. *Let C be a Jordan curve composed of a finite number of Jordan arcs meeting in corners* $z_1, z_2, \cdots, z_r$, *of exterior openings* $\mu_1\pi, \mu_2\pi, \cdots, \mu_r\pi$, $(2 > \mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_r > 0)$, *and let the difference quotient of the mapping function* [*see* (2)] *be bounded in modulus on each sub-arc not containing a corner. Let* $t = \mu_1$ *if* $\mu_1 \geqq 1$, *and* $t = 1$ *if* $\mu_1 < 1$. *Then we shall say that C is a curve of Type t.*

If C is a curve of Type t, the inequality $|P_n(z)| \leqq M$, *z on C, implies* $|P_n'(z)| \leqq MM_1 n^t$, *z on C* (see SII).

If C is a rectifiable Jordan curve it is shown in SI *that* $|P_n(z)| \leqq M$, *z on C, implies* $|P_n'(z)| \leqq MM_1 n^2$, *z on C.*

3. *Integral and Tchebycheff Approximation.* For approximation in the sense of least *p*th powers we have the following theorem.

THEOREM 1. *Let C be a rectifiable Jordan curve in the z-plane and let f(z) be analytic in C and continuous in* $\overline{C}$. *Let* $P_n(z)$, $(n = 1, 2, \cdots)$, *be a polynomial of degree n which minimizes*

$$\int_C w(z) \left| f(z) - P_n(z) \right|^p \left| dz \right|,$$

where w(z) is a bounded positive measurable function, with a positive lower bound, and p is a fixed positive number. Let $p_n(z)$ *be a polynomial of degree n such that* $|f(z) - p_n(z)| \leqq \epsilon_n$, *z in* $\overline{C}$. *Then we have*

$$|f(z) - P_n(z)| \leqq M n^{t/p} \epsilon_n, \qquad (z \text{ in } \overline{C}), \tag{3}$$

where M is a constant independent of n and z, and $t = 1$ *if C is a curve of Type S,* $1 \leqq t < 2$ *if C is a curve of Type t, and* $t = 2$ *if C is an arbitrary rectifiable Jordan curve.*

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* $f'(z)$ denotes the first derivative of $f(z)$.

We indicate the proof (see JI). Let $r_n(z)=f(z)-p_n(z)$, and let $\pi_n(z)=P_n(z)-p_n(z)$; then $r_n(z)-\pi_n(z)=f(z)-P_n(z)$. Let

$$\gamma_n = \int_C w(z)\,|\,f(z) - P_n(z)\,|^p\,|\,dz\,|$$

$$= \int_C w(z)\,|\,r_n(z) - \pi_n(z)\,|^p\,|\,dz\,|.$$

Let $|\pi_n(z)| \leqq \mu_n$, z on C, and let $|\pi_n(z_0)| = \mu_n$, where z_0 is a point of C. Then by the results of §2 we know that $|\pi_n'(z)| \leqq \mu_n M_1 n^t$, z on C, and it follows that

$$|\,\pi_n(z) - \pi_n(z_0)\,| \leqq \mu_n M_1 n^t\,|\,z - z_0\,|,$$

z on C. Let s be the arc or arcs of C consisting of the set of points ζ: ζ on C, $|\zeta - z_0| < 1/(2M_1 n^t)$. On s we have $|\pi_n(\zeta) - \pi_n(z_0)| \leqq \mu_n/2$. This means that on s, *whose total length is not less than* $1/(M_1 n^t)$, we have $|\pi_n(\zeta)| \geqq \mu_n/2$. Let $V \geqq w(z) \geqq v > 0$, and we have for $\mu_n \geqq 4\epsilon_n$

$$\gamma_n \geqq v\left(\frac{\mu_n}{4}\right)^p \frac{1}{M_1 n^t}, \tag{4}$$

and by the minimizing property of $P_n(z)$, we know that

$$\gamma_n \leqq LV\epsilon_n^p, \tag{5}$$

where L is the length of C. Combining inequalities (4) and (5) yields $\mu_n \leqq M_2 n^{t/p}\epsilon_n$, where M_2 is a constant depending on C, p, and $w(z)$, but independent of n and z. Thus whether $\mu_n \geqq \epsilon_n$ or not we have

$$|\,f(z) - P_n(z)\,| = |\,r_n(z) - \pi_n(z)\,| \leqq M n^{t/p}\epsilon_n, \qquad (z \text{ in } \overline{C}),$$

where M is independent of n and z, and the proof is complete.

Jackson (JI) establishes Theorem 1 with $t=1$ for C satisfying the condition that there is a number $r_0 > 0$ such that at every point of C a circle of radius r_0 can be drawn tangent to C, and containing in its interior and on its boundary only points of $\overline{C}$.* He also obtains (JII) the result with $t=2$ for the case where C is a curve such that its parametric representation satisfies a

* For a discussion of such properties see Seidel, loc. cit.

Lipschitz condition* of order 1 and R is a region for which there is a positive number r_0 such that from every point of its boundary a line segment of length $2r_0$ can be drawn belonging wholly to the closed region $\overline{R}$.

If C is an analytic Jordan curve and $f^{(j)}(z)$, $(j \geqq 0)$, satisfies† a Lipschitz condition of order α, $(0 < \alpha \leqq 1)$, on $\overline{C}$, we know by a theorem of John Curtiss‡ that $p_n(z)$ exists such that $\epsilon_n \leqq M_1/n^{j+\alpha}$, where M_1 is a constant independent of n and z. Consequently, in this case we have by inequality (3)

$$|f(z) - P_n(z)| \leqq \frac{M}{n^{j+\alpha-1/p}}, \qquad (z \text{ in } \overline{C}); \tag{6}$$

and hence if $j+\alpha > 1/p$, we have uniform convergence of the sequence $P_n(z)$ to $f(z)$ in $\overline{C}$. In fact (6) gives an upper bound on the degree of Tchebycheff approximation of $P_n(z)$.

In connection with (3) it is interesting to note that for each n we have§

$$\lim_{p \to \infty} P_n(z) = T_n(z),$$

where $P_n(z)$ is the polynomial of degree n of best approximation to the continuous function $f(z)$ on C, a rectifiable Jordan curve, in the sense of least pth powers with a norm function,‖ and $T_n(z)$ is the polynomial of degree n of best approximation to $f(z)$ on C in the sense¶ of Tchebycheff.

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* $f(z)$ satisfies a Lipschitz condition of order α on the set E if for arbitrary points z_1 and z_2 on E we have $|f(z_1) - f(z_2)| \leqq L|z_1 - z_2|^\alpha$, where L is a constant independent of z_1 and z_2.

† $f^{(0)}(z) \equiv f(z)$.

‡ *A note on the degree of polynomial approximation*, this Bulletin, vol. 42 (1936), pp. 873–878.

§ G. Julia, *Sur les polynomes de Tchebycheff*, Comptes Rendus (Paris), vol. 182 (1926), pp. 1201–1202; the corresponding result for C the segment (0, 1) of the axis of reals is due to G. Pólya, *Sur un algorithme toujours convergent pour obtenir les polynomes de meilleure approximation de Tchebycheff pour une fonction continue quelconque*, ibid., vol. 153 (1915), pp. 840–843.

‖ See J. L. Walsh, *Approximation by Polynomials in the Complex Domain*, Mémorial des Sciences Mathématiques, vol. 73 (1935); especially p. 24.

¶ That is, $\max [|f(z) - T_n(z)|, z \text{ on } C]$ is less than the corresponding expression for any other polynomial of degree n.

4. *The Case** $p=2$. Approximation in the sense of least squares leads to a consideration of the set of polynomials $\{P_n(z)\}$ normal and orthogonal on C, a rectifiable Jordan curve in the z-plane. The method used in proving Theorem 1 serves to establish the following result.

THEOREM 2. *Let C be a rectifiable Jordan curve in the z-plane and let $\int_C |Q_n(z)|^p |dz| = \epsilon_n$, $p>0$, where $Q_n(z)$ is a polynomial of degree n. Then we have*

$$|Q_n(z)| \leqq M n^{t/p} \epsilon_n^{1/p}, \qquad (z \text{ on } C), \tag{7}$$

where M is a constant independent of n and z, and $t=1$ if C is a curve of Type S, $1 \leqq t < 2$ if C is a curve of Type t, and $t=2$ if C is an arbitrary rectifiable Jordan curve.

For the set of polynomials $\{P_n(z)\}$ normal and orthogonal on C, we have $\epsilon_n = 1$ with $p=2$ in Theorem 2. Thus (7) becomes

$$|P_n(z)| \leqq M n^{t/2}, \qquad (z \text{ on } C).$$

Now suppose $f(z)$ belongs to L^2 on C; then

$$f(z) \sim a_0 P_0(z) + a_1 P_1(z) + \cdots + a_k P_k(z) + \cdots,$$

$$a_k = \int_C f(z) \overline{P_k(z)} \, |dz|,$$

where the sign $\sim$ is used to denote formal correspondence. The polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares is $S_n(z) = \sum_{k=0}^{n} a_k P_k(z)$. Now suppose $\sum_{k=0}^{\infty} |a_k| k^{t/2}$, where the value of t is compatible with the character of C, converges, then

$$f_1(z) = \sum_{k=0}^{\infty} a_k P_k(z)$$

converges absolutely and uniformly on, and hence within, C and consequently $f_1(z)$ is analytic in C and continuous in $\overline{C}$. Moreover, since $S_n(z)$ converges in the mean to $f(z)$ on C, the function $f_1(z) = f(z)$ almost everywhere on C. Thus we have the following theorem.

* See G. Szegö, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, Mathematische Zeitschrift, vol. 9 (1921), pp. 218–270; J. L. Walsh, *Interpolation and Approximation*, Chap. VI.

THEOREM 3. *Let C be a rectifiable Jordan curve in the z-plane and let $f(z)$ belong to L^2 on C. Let $\{P_n(z)\}$ be the set of polynomials normal and orthogonal on C and let*

$$f_1(z) = a_0P_0(z) + a_1P_1(z) + \cdots + a_kP_k(z) + \cdots,$$

$$a_k = \int_C f(z)\overline{P_k(z)}\,|\,dz\,|.$$

Suppose $\sum_{k=0}^{\infty}|a_k|k^{t/2}$, $(2 \geqq t \geqq 1)$, converges. Then $f_1(z)$ is analytic in C, continuous in $\overline{C}$, and is equal to $f(z)$ almost everywhere on C if either: (1) *C is a curve of Type S and $t=1$, or* (2) *C is curve of Type t and $2>t\geqq 1$, or* (3) *C is an arbitrary rectifiable Jordan curve and $t=2$.*

Of course in the above theorem the function $f(z)$ may be defined (or redefined) on a set of measure 0 on C and defined in C so as to coincide everywhere with $f_1(z)$.

GEORGIA SCHOOL OF TECHNOLOGY

EQUIVALENCE OF ALGEBRAIC EXTENSIONS†

BY REINHOLD BAER

The commutative fields‡ K and H are *equivalent* with regard to their common subfield L, if there exists an isomorphism between K and H which maps every element of L upon itself. If H and K are equivalent with regard to L, then the same equations with coefficients in L have solutions in H and in K. It is the aim of this note to establish a criterion for the validity of the converse of the above proposition.

The field F is *completely algebraic* with regard to its subfield S, if F and S satisfy:

(1) *F is algebraic with regard to S;*

(2) *if $\mathfrak{f}$ is an isomorphism of S upon the subfield S' of the field G' such that every equation (with coefficients) in S which has a solution in F is mapped by $\mathfrak{f}$ upon an equation in S' which has a solution in G', then $\mathfrak{f}$ is induced by an isomorphism of F upon a field F' between S' and G' ($S' \leqq F' \leqq G'$).*

E. Steinitz§ has proved that every simple algebraic extension‖ and every normal algebraic extension¶ is completely algebraic.

LEMMA 1. *If the algebraic extension F of the field S satisfies the condition* (i) *to every pair of fields U and V such that $S \leqq U \leqq V \leqq F$, V finite with regard to U, there exists a field W between V and F such that W is finite and completely algebraic with regard to U, then F is completely algebraic with regard to S.*

PROOF. There exists a chain of fields F_v (v an ordinal number

† Presented to the Society, October 31, 1936.

‡ Only commutative fields will be considered in this note.

§ See E. Steinitz, *Algebraische Theorie der Körper; Mit Erläuterungen und einem Anhang: Abriss der Galois-schen Theorie*, neu herausgegeben von Reinhold Baer und Helmut Hasse, 1931.

‖ S is the simple algebraic extension of the field F, generated by the element b, if b satisfies an algebraic equation with coefficients in F and S is a smallest field containing F and b.

¶ N is normal with regard to its subfield S if every irreducible polynomial in S which has zeros in N is in N a product of linear polynomials.

taking all values between 0 and a certain ordinal number k) such that $F_0=S$, $F_{v+1}=F_v(e_v)$ is a simple (algebraic) extension of F_v for $0\leqq v\leqq k$, F_v is the join of all the F_u with $0\leqq u<v$, if v is a limit ordinal, $F_k=F$. This chain of fields F_v satisfies in particular the relations $S\leqq F_u\leqq F_v\leqq F$, if $u<v$.

Assume now that G' is an algebraic extension of the field S' and that there exists an isomorphism $\mathfrak{s}$ of S upon S' such that (I) every equation in S with solutions in F is mapped by $\mathfrak{s}$ upon an equation in S' with solutions in G'.

There will now be constructed by complete (transfinite) induction a chain of fields F_v' and of isomorphisms $\mathfrak{f}_v$ of F_v upon F_v' such that $S'\leqq F_u'\leqq F_v'\leqq G'$ for $u<v$; $\mathfrak{f}_0=\mathfrak{s}$, $\mathfrak{f}_v$ induces $\mathfrak{f}_u$ in F_u for $u<v$; every $\mathfrak{f}_v$ satisfies (I). Since $F_0'=S'$ and $\mathfrak{f}_0=\mathfrak{s}$ is a suitable beginning for this construction, it may be assumed that F_v', $\mathfrak{f}_v$ have been defined for every v with $0\leqq v<u$ ($\leqq k$) and that they satisfy the above conditions.

CASE 1. *$u=w+1$ is not a limit ordinal.*

Denote by $g(x)$ the irreducible polynomial in F_w whose zero is e_w. It is mapped by $\mathfrak{f}_w$ upon a certain polynomial $g^*(x)$ in F_w' and $g^*(x)$ has zeros in G', since $\mathfrak{f}_w$ satisfies (I). Let $b_1, \cdots, b_n$ be the set of all the zeros of $g^*(x)$ in G'; n is finite and positive. Since $g(x)$ is irreducible in F_w, $\mathfrak{f}_w$ is an isomorphism, it follows that $g^*(x)$ is irreducible in F_w', and consequently there exists for every i exactly one isomorphism $\mathfrak{h}_i$ which induces $\mathfrak{f}_w$ in F_w and maps F_u upon $F_w'(b_i)$ and e_u upon b_i.

If $\mathfrak{h}_i$ does not satisfy (I), then there exists a polynomial $q_i(x)$ in F_u which has zeros in F such that the polynomial $q_i^*(x)$ upon which $q_i(x)$ is mapped by $\mathfrak{h}_i$ has no zero in G'. Assume that none of the isomorphisms $\mathfrak{h}_i$ satisfies (I). The set of all the solutions in F of the equation $g(x)q_1(x)\cdots q_n(x)=0$ generates a certain field V between F_w and F which contains F_u and is finite with regard to F_w. There exists therefore by condition (i) a field W between V and F which is finite and completely algebraic with regard to F_w. Since the isomorphism $\mathfrak{f}_w$ of the subfield F_w of W satisfies (I), it follows that there exists an isomorphism $\mathfrak{h}$ of W upon a field W' between F_w' and G' which induces $\mathfrak{f}_w$ in F_w. Since the isomorphisms $\mathfrak{h}_1, \cdots, \mathfrak{h}_n$ are all the possible isomorphisms of the subfield F_u upon a field between F_w' and G' which induce $\mathfrak{f}_w$ in F_w, and since F_u is a subfield of W, $\mathfrak{h}$ induces in F_u exactly

one of the isomorphisms $\mathfrak{h}_i$, say $\mathfrak{h}_1$. Since $q_1(x)$ has zeros in W, $q_1(x)$ is mapped by $\mathfrak{h}$ and by $\mathfrak{h}_1$ upon a polynomial in $F_w'(b_1)$ which has zeros in G' in contradiction to the above assumption. Therefore at least one of the isomorphisms $\mathfrak{h}_i$, say $\mathfrak{h}_n$, satisfies (I) and it may be defined by $\mathfrak{f}_u = \mathfrak{h}_n$, $F_u' = F_w'(b_n)$, and thus the chain of fields and isomorphisms has been prolongated in the required way.

CASE 2. *u is a limit ordinal.*

Then F_u is the join of the increasing chain of fields F_v with $v<u$. If F_u' is defined as the join of the increasing chain of fields F_v' with $v<u$, then there exists one and only one isomorphism† $\mathfrak{f}_u$ of F_u upon F_u' which induces $\mathfrak{f}_v$ in F_v for $v<u$. Since every polynomial in F_u is contained in a certain F_v with $v<u$, and since every $\mathfrak{f}_v$ with $v<u$ satisfies (I), it follows that also $\mathfrak{f}_u$ satisfies (I).

Thus it has been proved that fields F_u' and isomorphisms $\mathfrak{f}_u$, satisfying the above conditions, exist for every u with $0 \leqq u \leqq k$. There exists therefore, in particular, an isomorphism $\mathfrak{f}_k$ of $F_k = F$ upon a field F_k' between S' and G' which induces the given isomorphism $\mathfrak{s}$ in S, and therefore F is completely algebraic with regard to S.

COROLLARY 1. *If F is algebraic and separable‡ with regard to its subfield S, then F is completely algebraic with regard to S.*

PROOF. If $S \leqq U \leqq V \leqq F$, then F and V are separable with regard to U. V is therefore a simple algebraic extension of U if, and only if, V is finite with regard to U. Since simple algebraic extensions are completely algebraic, it follows therefore that F satisfies (i) with regard to S, and consequently Lemma 1 implies that F is completely algebraic with regard to S.

THEOREM 1. *Assume that F is completely algebraic with regard to its subfield S, that F' is algebraic with regard to its subfield S', and that $\mathfrak{s}$ is an isomorphism of S upon S'. Then there exists an*

† See E. Steinitz, loc. cit., *Erläuterungen*, pp. 17–18.

‡ F is separable (and algebraic) with regard to its subfield S, if every element of F is a solution of a separable irreducible equation in S. An irreducible equation $g(x)=0$ in S is not separable if the characteristic of S is a prime number p and $g(x)=\sum_{i=0}^{n} s_i x^{pi}$. All the solutions of a separable equation are *different*. Every finite separable extension is simple.

isomorphism of F upon F' which induces $\mathfrak{s}$ in S if, and only if, every polynomial $g(x)$ in S has the same number of zeros in F as the polynomial $g(x)\mathfrak{s}=g^*(x)$ (upon which $g(x)$ is mapped by $\mathfrak{s}$) has zeros in F'.

PROOF. Clearly it suffices to prove that the condition is sufficient. From the assumptions it follows that there exists an isomorphism $\mathfrak{f}$ of F upon a field F^* between S' and F' which induces $\mathfrak{s}$ in S. Since $g(x)$ has as many zeros in F as $g(x)\mathfrak{f}=g(x)\mathfrak{s}$ has zeros in F^* and in F', it follows that all the elements of F' which are algebraic with regard to S' are contained in F^*, and since F' is algebraic with regard to S', it follows that $F'=F^*$, that is that $\mathfrak{s}$ is induced by the isomorphism $\mathfrak{f}$ of F upon F'.

COROLLARY 2. *If F is algebraic and separable with regard to its subfield S, and F' is algebraic with regard to its subfield S', then the isomorphism $\mathfrak{s}$ of S upon S' is induced by an isomorphism of F upon F' if, and only if, the polynomial $g(x)$ in S has a zero in F if, and only if, $g(x)\mathfrak{s}$ has a zero in F'.*

PROOF. It suffices to prove the sufficiency of the condition. Since an algebraic extension of a field is separable if, and only if, the solved irreducible equations of the subfield are separable,† F' is separable with regard to S'. By Corollary 1, F and F' are completely algebraic with regard to S and S', respectively. There exists therefore

an isomorphism $\mathfrak{f}$ of F upon a field G' between S' and F' which induces $\mathfrak{s}$ in S.	an isomorphism $\mathfrak{g}$ of F' upon a field G between S and F which induces $\mathfrak{s}^{-1}$ in S'.

If the polynomial $g(x)$ in S has n zeros in F and the polynomial $g(x)\mathfrak{s}$ has n' zeros in F', then $n\leqq n'\leqq n$, that is, $n=n'$, since the zeros of $g(x)$ are mapped by $\mathfrak{f}$ upon zeros of $g(x)\mathfrak{s}$ and the zeros of $g(x)\mathfrak{s}$ are mapped by $\mathfrak{g}$ upon zeros of $g(x)$. The isomorphism $\mathfrak{s}$ satisfies therefore the condition of Theorem 1 and is consequently induced by an isomorphism of F upon F'.

COROLLARY 3. *If F and G are algebraic with regard to their common subfield S, and if F is separable with regard to S, then F and G are equivalent with regard to S if, and only if, the same equations in S have solutions in F and in G.*

† See Steinitz, loc. cit.

Suppose that F is a field of characteristic $p \neq 0$ and that K is a subfield of F. Then the element x of F is said to be a *root* with regard to K if there exists an integer $n = n(x) \geqq 0$ such that x^{p^n} is an element of K. The set $R(K < F)$ of all the elements in F which are roots with regard to K is a subfield of F, containing K.†

LEMMA 2. *Suppose that K_i is a subfield of the field F_i of characteristic $p \neq 0$, and that there exists an isomorphism $\mathfrak{k}$ of K_1 upon K_2, satisfying the condition* (K) *an equation with coefficients in K_1 has a solution in F_1 if, and only if, the (under $\mathfrak{k}$) corresponding equation in K_2 has a solution in F_2. Then there exists one and only one isomorphism $\mathfrak{r}$ of $R(K_1 < F_1)$ upon $R(K_2 < F_2)$ which induces $\mathfrak{k}$ in K_1 and this isomorphism $\mathfrak{r}$ satisfies the condition* (R) *an equation with coefficients in $R(K_1 < F_1)$ has a solution in F_1 if, and only if, the (under $\mathfrak{r}$) corresponding equation in $R(K_2 < F_2)$ has a solution in F_2.*

PROOF. If the element b of F_i is a root with regard to K_i, then there exists a smallest not negative integer $e = e(b)$ such that $b^{p^{e(b)}} = b^*$ is an element of K_i. The polynomial‡ $f_b(x) = x^{p^{e(b)}} - b^*$ is irreducible in K_i and its only zero in F_i is b, since $f_b(x) = (x-b)^{p^{e(b)}}$ in F_i. Conversely, a polynomial of the form $x^{p^j} - c$ in K_i has at most one zero in F_i.

If now b is any element of $R(K_1 < F_1)$, then it follows from these remarks and from condition (K) that there exists exactly one zero of the polynomial $x^{p^{e(b)}} - b^*\mathfrak{k}$ in F_2, and this uniquely determined element of F_2 may be denoted by $b\mathfrak{r}$. It follows from the mentioned properties of the fields $R(K_i < F_i)$ and from the equations $(x+y)^{p^j} = x^{p^j} + y^{p^j}$, that $\mathfrak{r}$ is an isomorphism of $R(K_1 < F_1)$ upon $R(K_2 < F_2)$ which induces $\mathfrak{k}$ in K_1, and that $\mathfrak{r}$ is the only isomorphism with this property.

From the formula which has just been mentioned it follows that there exists to every polynomial $f(x)$ in $R(K_i < F_i)$ a not negative integer n such that $f(x)^{p^n}$ is a polynomial in K_i. The polynomial $f(x) = 0$ has therefore a solution in F_i if, and only if, the equation $f(x)^{p^n} = 0$ (with coefficients in K_i) has a solution in F_i. Now (R) is a consequence of (K).

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† See Steinitz, loc. cit., §§11–14.

‡ See Steinitz, loc. cit., §§11–14.

A consequence of this Lemma 2 and of Corollary 2 is the following corollary.

COROLLARY 4. *Suppose that the field F_1 of characteristic $p \neq 0$ is algebraic with regard to its subfield K and that F_1 is separable with regard to $R(K < F_1)$. Then F_1 and F_2 are equivalent extensions of their common subfield K if, and only if, the same equations in K have solutions in F_1 and in F_2.*

REMARK. If F is an algebraic extension of the field K of characteristic $p \neq 0$, then it may happen that F is *not* separable with regard to $R(K < F)$.† If F is algebraic and normal with regard to K, then F is always‡ separable with regard to $R(K < F)$. Corollary 4 contains therefore the analogous proposition concerning normal algebraic extensions which has been mentioned in the beginning of this note.

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† An example for this phenomenon may be found in R. Baer, *Abbildungseigenschaften algebraischer Erweiterungen*, Mathematische Zeitschrift, vol. 33 (1931), pp. 451–479, particularly pp. 471–472.

‡ See Baer, loc. cit., Satz 14 on p. 471; for further criteria see Sätze 15 and 16.

A COMPLETE SYSTEM FOR THE SIMPLE GROUP G_{60}^6

BY C. W. STROM

The problem of this paper is to obtain an irreducible set of polynomials in terms of which all the polynomials that are invariant under the simple group G_{60}^6 can be expressed as polynomial functions.

No polynomials of degrees and extents 1 and 2 respectively are invariant under this group except the respective elementary symmetric polynomials, E_1 and E_2. Hence, this irreducible set will contain no polynomials of extents 1 and 2 other than the elementary symmetric polynomials which we shall write as S-polynomials:

$$S_1 \equiv E_1 \quad \text{and} \quad S_2 \equiv E_2.$$

The group leaves invariant the set of triples,

123, 134, 145, 156, 162, 235, 346, 452, 563, 624,

and the complementary set

124, 146, 163, 135, 152, 243, 465, 632, 354, 526.

Hence the S-polynomials,

$$S_{123} \equiv x_1x_2x_3 + x_1x_3x_4 + x_1x_4x_5 + \cdots,$$
$$S_{124} \equiv x_1x_2x_4 + x_1x_4x_6 + x_1x_6x_3 + \cdots,$$

or, as we shall write them,

$$S_{123} \equiv 123 + 134 + 145 + \cdots,$$
$$S_{124} \equiv 124 + 146 + 163 + \cdots,$$

are invariant under the group. Clearly, the polynomials $S_{1^{i_2}2^{i_3}3^{i_4}}$ and $S_{1^{i_2}2^{i_3}4^{i_4}}$ are also invariant under the group, where

$$S_{1^{i_2}2^{i_3}3^{i_4}} + S_{1^{i_2}2^{i_3}4^{i_4}} \equiv \Sigma_{1^{i_2}2^{i_3}3^{i_4}},$$

the general symmetric polynomial of extent 3 on 6 variables.

Each of the 15 quadruples that can be selected from among the numbers $1, \cdots, 6$ may be regarded as the *intersection* of two of the triples in each of the sets above. Thus, 1234 is the

intersection of 123 and 134 in the first set and of 124 and 234 in the second set. We shall write the quadruple 1234 in the form $13\cdot 24$ and we shall define the S-polynomial of extent 4 and degree 4:

$$\begin{aligned} S_{13\cdot 24} \equiv\ & 13\cdot 24 + 23\cdot 45 + 12\cdot 36 + 45\cdot 12 + 26\cdot 14 + 16\cdot 25 \\ & + 14\cdot 35 + 34\cdot 16 + 56\cdot 13 + 15\cdot 46 + 25\cdot 34 + 35\cdot 26 \\ & + 24\cdot 56 + 36\cdot 45 + 46\cdot 23. \end{aligned}$$

Then, $S_{13\cdot 24} \equiv E_4$, but

$$\begin{aligned} \Sigma_{1^i 2^j 3^k 4^l} \equiv\ & S_{1^i 3^j\cdot 2^k 4^l} + S_{1^j 3^k\cdot 2^i 4^l} + S_{1^i 3^j\cdot 2^l 4^k} \\ & + S_{1^i 3^k\cdot 2^l 4^j} + S_{1^i 3^l\cdot 2^k 4^j} + S_{1^k 3^l\cdot 2^i 4^j}. \end{aligned}$$

Each of the six quintuples that can be selected from among the numbers $1, \cdots, 6$ contains ten triples and these form five pairs, consisting of one triple from each set having an element in common. Thus, the quintuple 12345 contains the following pairs:

513, 524; 423, 415; 345, 321; 214, 253; and 125, 143.

We define the S-polynomial of extent 5 and degree 5: $S_{5\cdot 13\cdot 24} \equiv 5\cdot 13\cdot 24 + 4\cdot 23\cdot 15 + 3\cdot 45\cdot 32 + 2\cdot 14\cdot 53 + 1\cdot 25\cdot 43$ + terms obtained from the five remaining quintuples. We define further

$$\begin{aligned} S_{5^i\cdot 1^j 3^k\cdot 2^l 4^m} \equiv\ & 5^i\cdot 1^j 3^k\cdot 2^l 4^m + 5^i\cdot 1^k 3^j\cdot 2^m 4^l + 4^i\cdot 2^j 3^k\cdot 1^l 5^m \\ & + 4^i\cdot 2^k 3^j\cdot 1^m 5^l + 3^i\cdot 4^j 5^k\cdot 2^l 1^m + 3^i\cdot 4^k 5^j\cdot 2^m 1^l \\ & + 2^i\cdot 1^j 4^k\cdot 5^l 3^m + 2^i\cdot 1^k 4^j\cdot 5^m 3^l + 1^i\cdot 2^j 5^k\cdot 4^l 3^m \\ & + 1^i\cdot 2^k 5^j\cdot 4^m 3^l + 50 \text{ additional terms,} \end{aligned}$$

where the additional terms are obtained from the 5 remaining quintuples. Then

$$S_{5\cdot 13\cdot 24} \equiv 5E_5,$$

and

$$\begin{aligned} 5\Sigma_{1^i 2^j 3^k 4^l 5^m} \equiv\ & S_{5^i\cdot 1^j 3^k\cdot 2^l 4^m} + S_{5^i\cdot 1^j 3^k\cdot 2^m 4^l} + S_{5^i\cdot 1^j 3^l\cdot 2^k 4^m} \\ & + S_{5^i\cdot 1^j 3^l\cdot 2^m 4^k} + S_{5^i\cdot 1^j 3^m\cdot 2^k 4^l} + S_{5^i\cdot 1^j 3^m\cdot 2^l 4^k} + S_{5^i\cdot 1^k 3^j\cdot 2^l 4^m} \\ & + S_{5^i\cdot 1^k 3^j\cdot 2^m 4^l} + S_{5^i\cdot 1^k 3^m\cdot 2^j 4^l} + S_{5^i\cdot 1^k 3^m\cdot 2^l 4^j} + S_{5^i\cdot 1^l 3^m\cdot 2^j 4^k} \\ & + S_{5^i\cdot 1^l 3^m\cdot 2^k 4^j} + 48 \text{ additional } S\text{–polynomials,} \end{aligned}$$

where the additional polynomials are obtained by placing j, k, l, and m in succession as the exponent on the outstanding element, 5.

It is to be noted that each S-polynomial of extent 5 can be written in five different ways. Thus

$$S_{5^j \cdot 1^i 3^k \cdot 2^l 4^m} \equiv S_{4^m \cdot 2^l 3^k \cdot 1^i 5^j} \equiv S_{3^k \cdot 4^m 5^j \cdot 2^l 1^i} \equiv S_{2^l \cdot 1^i 4^m \cdot 5^j 3^k} \equiv S_{1^i \cdot 2^l 5^j \cdot 4^m 3^k}.$$

The general symmetric polynomial of extent 6 consists of the sum of S-polynomials of extent 6, but since each of these is the product of some power of the elementary symmetric polynomial of degree 6 by some S-polynomial of extent less than 6, they do not require further examination.

It is clear from their definitions that all the S-polynomials defined above belong to the simple group G_{60}^6 and that every polynomial that belongs to this group is a polynomial in the S-polynomials. The irreducible set that we seek is therefore to be selected from among the S-polynomials.

Since we may take $x_1, \cdots, x_6$ as the roots of the equation

$$(x - x_1)(x - x_2) \cdots (x - x_6) = 0,$$

we can express powers of $x_i \geqq 6$ in terms of the elementary symmetric polynomials and powers of $x_i < 6$. We may therefore confine our attention to S-polynomials for which $1 \leqq i, j, \cdots, k \leqq 6$.

The general method of procedure differs only in details from that employed in investigating the complete system of the simple group G_{168}^7 which was described in an earlier paper.* This procedure isolates a small set of polynomials from which the following 11, forming an irreducible set, may be conveniently chosen as a complete system for the polynomials that are invariant under the simple group G_{60}^6:

$$E_i, \quad (i = 1, 2, \cdots, 6), \qquad S_{123}, \quad S_{1^j 3 \cdot 2^i 4}, \quad (j = 2, 3, 4, 5).$$

VANCOUVER, B. C.

* *On complete systems under certain finite groups*, this Bulletin, vol. 37 (1931), p. 570.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STATE COLLEGE, PENNSYLVANIA, SUMMER MEETING AND COLLOQUIUM, September 7–10, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than August 3. Abstracts received by the Secretaries not later than July 9 will appear in the July issue of the Bulletin. Professor John von Neumann will deliver a series of colloquium lectures on *Continuous geometry*. By invitation of the Program Committee, Professor Hassler Whitney will give an address entitled *Topological properties of differentiable manifolds*.

NEW YORK CITY, October 30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than October 2. Abstracts received by the Secretaries not later than September 3 will appear in the September issue of the Bulletin. By invitation of the Program Committee, Professor D. J. Struik will deliver an address entitled *The application of tensor analysis to problems of electrical engineering*.

IOWA CITY, IOWA, November 26–27, 1937.

Abstracts are due by October 30, 1937.

PASADENA, CALIFORNIA, November 27, 1937.

Abstracts are due by October 30, 1937.

INDIANAPOLIS, INDIANA, ANNUAL MEETING, December 28–30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than November 23, 1937. Abstracts received by the secretaries not later than November 2 will appear in the November issue of the Bulletin.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to the Managing Editor, D. R. CURTISS, Northwestern University, Evanston, Ill. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1936, p. 67) are payable to the Treasurer of the Society, Professor P. A. Smith, Low Memorial Library, 531 West 116th St., New York, N. Y.

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